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TRANSLATION

CERTAIN MATHEMATICAL METHODS OF SOLVING ENGINEERING PROBLEMS

By

I. A. Birger

FOREIGN TECHNOLOGY DIVISION

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CERTAIN MATHEMATICAL METHODS OF SOLVING ENGINEERING PROBLEMS

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TABLE OF CONTENTS

In book there is given an application of normal fundamental functions and integral equations for solving engineering problems.

Examples, considered in work, refer to problems of strength, stability and vibrations of elastic systems, however, the results can be used also in other fields of engineering.

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INTRODUCTION

Frequently the solution of engineering problems reduces to a solution of ordinary differential equations or their systems with boundary conditions of a general form. If the corresponding equation has a high order and variable coefficients, then the problem is found to be difficult, since the finding of an accurate solution usually is not successful. This refers even to second order equations, if they do not reduce to known equations (for example, Bessel equation), the solutions of which have been tabulated.

Difficulties arise also in those cases, when an accurate solution is known (for example, for differential equations with constant coefficients), but within limits of interval of changes of the independent variable, the sought function or its derivatives experience discontinuities (for example, problem on flexure of rod under action of concentrated forces and moments, problem on distribution of temperature in rods with branches).

The most effective way of solving in the latter case is the application of normal fundamental functions, as was demonstrated in works of the outstanding Academician, mechanics and mathematics specialist, A. N. Krylov.

In those cases, when the problem reduces to differential equations with variable coefficients, it seems expedient to proceed to integral equations. The idea of such a transition is intimately connected with application of method of

successive approximations for the solution of differential equations, however, a transition to integral equations makes it possible to use more general and more effective solutions.

In this work there is considered the application of method of normal fundamental functions (Chapters 1 and 2), and also there are investigated boundary and normal integral equations (Chapters 3 and 4).

Examples of application refer to problems of Engineering Mechanics, however the fairly general discussion makes it possible to apply the results also in solving other engineering problems.

Author expresses gratitude to Academician L. I. Sedov, to Acting Member of Academy of Sciences of Ukrainian SSR, S. V. Serensen, Professors F. R. Gantmakher, R. S. Kinasoshvili, S. D. Ponomarev, P. M. Riz, Doctors of Engineering Sciences, V. K. Zhitomirskiy, V. Ya. Natanzon for critical remarks and advices in reviewing the manuscript.

CHAPTER 1

NORMAL FUNDAMENTAL FUNCTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Effectiveness of use of normal fundamental functions in engineering problems was established in widely known works of A. N. Krylov. In the subsequent works of P. F. Papkovich, Sh. E. Mikeladze, N. K. Snitko and others, these functions were applied in solving a number of problems of structural mechanics.

The special advantages of normal fundamental functions are reflected in constructing discontinuous solutions of differential equations with constant coefficients (of solutions with given discontinuities of the derivatives).

Comprehensive experience in constructing and using such solutions in problems of structural mechanics (beginning with the known problem of integrating equation of elastic line of rod) made it possible to generalize the results for linear differential equations of arbitrary structure (Works of Sh. E. Mikeladze).

In this work there are established general formulas for determining normal fundamental functions and simple differential relationships between them.

By means of general relationships there are obtained already well-known systems of normal fundamental functions and there are given certain applications of these functions. In particular, they are used for an approximate integration of differential equations with variable coefficients.

The solution of differential equation is presented in matrix form, which is a mathematical expression of a known method in structural mechanics of initial

parameters (Works of A. A. Umanskiy, N. I. Bezukhov, N. G. Chudnovskiy and others).

1. Statement of Problem

Suppose there is given a linear differential equation of n-th order

$$L(y)=y^{(n)}(x)+p_1(x)y^{(n-1)}(x)+\ldots+p_n(x)y(x)=f(x) \quad (1.1)$$

or

$$\sum_{i=0}^{n} p_i(x)y^{(n-i)}(x)=f(x)$$
$$[p_0(x)=1,\quad y^{(0)}(x)=y(x)]. \quad (1.2)$$

the solution of equation (1.1) in a certain interval of change $x(a<x<b)$, is sought.

The totality of n (linearly independent) solutions of the homogeneous equation (1.1) $\{Y_k(x)\}$ $(k=0,1,\ldots,n-1)$, satisfying the condition

$$Y_k^{(i)}(a)=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases} \quad (i,\ k=0,\ 1,\ \ldots,n-1), \quad (1.3)$$

is called the normal fundamental system of solutions of equation (1.1) with the initial section x = a.

The particular solution of equation (1.1), corresponding to zero initial conditions, is designated $Y_*(x)$.

Thus,

$$Y_*^{(i)}(a)=0 \quad (i=0,\ 1,\ \ldots,n-1). \quad (1.4)$$

(If $\{Y_k(x)\}$ and $Y_*(x)$, are known then the solution of equation (1.1) is presented as:

$$y(x)=\sum_{k=0}^{n-1} y^{(k)}(a)\,Y_k(x)+Y_*(x). \quad (1.5)$$

where $y^{(k)}(a)$ $(k=0,...,n-1)$ -- are values of the function y(x) and its first n--1 derivatives in the initial section x = a. The indicated values are called also initial parameters. In solving boundary value problems, there is used usually not only the function y(x), but also its derivative up to n--1 order inclusively.

For the future it is expedient to introduce "the column--solution"

$$[y(x)] = \begin{bmatrix} y(x) \\ y^{(1)}(x) \\ \cdots \\ y^{(n-1)}(x) \end{bmatrix}. \tag{1.6}$$

From equality (1.5) by successive differentiation we find

$$[y(x)] = [Y(x)]\,[y(a)] + [Y_*(x)], \tag{1.7}$$

where

$$[Y(x)] = \begin{bmatrix} Y_0(x) & Y_1(x) & \cdots & Y_{n-1}(x) \\ Y_0^{(1)}(x) & Y_1^{(1)}(x) & \cdots & Y_{n-1}^{(1)}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ Y_0^{(n-1)}(x) & Y_1^{(n-1)}(x) & \cdots & Y_{n-1}^{(n-1)}(x) \end{bmatrix} \tag{1.8}$$

is a normal fundamental matrix of the homogeneous equation (1.1), and

$$[y(a)] = \begin{bmatrix} y(a) \\ y^{(1)}(a) \\ \cdots \\ y^{(n-1)}(a) \end{bmatrix}, \quad [Y_*(x)] = \begin{bmatrix} Y_*(x) \\ Y_*^{(1)}(x) \\ \cdots \\ Y_*^{(n-1)}(x) \end{bmatrix}$$

is the column of initial parameters and the column of a particular solution. The normal fundamental matrix in initial section is unitary and column -- the particular solution -- is zero. The solution (1.7) corresponds to the application of

the method of initial parameters, widely used in engineering problems. This solution we shall call the <u>solution in matrix form</u>.

2. The Homogeneous Equation

Let us consider a homogeneous differential equation with constant coefficients:

$$\sum_{l=0}^{n} p_l y^{(n-l)}(x) = 0 \qquad (2.1)$$

$$(p_0 = 1,\ y^{(0)}(x) = y(x)).$$

Suppose $F(\lambda)$ is the characteristic polynomial of equation (2.1):

$$F(\lambda) = \sum_{l=0}^{n} p_l \lambda^{n-l}. \qquad (2.2)$$

the roots of which we shall designate λ_s $(s = 0, 1, \ldots)$.

If we subordinate the selection of arbitrary constants to the condition

$$y^{(l)}(a) = \eta^l \qquad (l = 0, 1, \ldots, n-1), \qquad (2.3)$$

where η is a certain parameter, then the solution of equation (2.1) will be such*

$$y(x) = B\, \frac{F(\lambda) - F(\eta)}{(\lambda - \eta) F(\lambda)}\, e^{\lambda(x-a)}, \qquad (2.4)$$

where B is the symbol of operation of a complete integral residue.

If now we expand expression (2.4) into a series by degrees of the parameter η, then, as was established even by Cauchy, the normal fundamental functions of equation (2.1) are found to be the coefficients of the expansion

$$y(x) = Y_0(x)\eta^0 + Y_1(x)\eta^1 + \ldots + Y_{n-1}(x)\eta^{n-1}. \qquad (2.5)$$

---

*A. N. Krylov, On Certain Differential Equations of Mathematical Physics, GITTL, M.--L., 1950.

After discussing this result A. N. Krylov in the work "On Certain Differential Equations of Mathematical Physics" turns to the consideration of concrete differential equations of simple structure, for which he also makes the indicated expansion. However, it is possible to establish certain general results, valid for differential equations of arbitrary order with constant coefficients.

We shall assume at first that the roots of the characteristic polynomial $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are all simple.

In introducing the value

$$\frac{F(\lambda)-F(\eta)}{\lambda-\eta}=\sum_{i=0}^{n-1} p_i(\lambda^{n-i-1}+\lambda^{n-i-2}\eta+\ldots+\eta^{n-i-1})=$$
$$=\sum_{k=0}^{n-1}\eta^k\sum_{i=0}^{n-1-k}p_i\lambda^{n-1-k-i} \tag{2.6}$$

into equality (2.4), by making a calculation of integral residue and by considering the expansion (2.5), we obtain*

$$Y_k(x)=\sum_{s=0}^{n-1}\frac{\sum\limits_{i=0}^{n-1-k}p_i\lambda_s^{n-1-k-i}}{\sum\limits_{i=0}^{n-1}p_i(n-i)\lambda_s^{n-i-1}}e^{\lambda_s(x-a)} \tag{2.7}$$
$$(k=0,1,\ldots,n-1;\quad p_0=1).$$

In this equality λ_s $(s=0,1,\ldots,n-1)$ are roots of the characteristic polynomial

In a particular case for function $Y_{n-1}(x)$ there is obtained the following expression:

$$Y_{n-1}(x)=\sum_{s=0}^{n-1}\frac{e^{\lambda_s(x-a)}}{\sum\limits_{i=0}^{n-1}p_i(n-i)\lambda_s^{n-i-1}}\qquad (p_0=1). \tag{2.8}$$

* This result also can be obtained by methods of operational calculus.

We now consider the case of multiple roots. Suppose the characteristic polynomial (2.2) has m different roots λ_s $(s=0, 1,\ldots,m-1)$ with a multiplicity ν_s.

Relationships (2.4) -- (2.6) remain in force also for the considered case and therefore

$$Y_k(x) = B\,\frac{\sum_{l=0}^{n-1-k} p_l\lambda^{n-1-k-l}}{F(\lambda)}\,e^{\lambda(x-a)}. \tag{2.9}$$

After calculating the complete integral residue, we will find

$$Y_k(x) = \sum_{s=0}^{m-1}\frac{1}{(\nu_s-1)!}\,\frac{\partial^{\nu_s-1}}{\partial\lambda^{\nu_s-1}}\left\{\frac{\sum_{l=0}^{n-1-k} p_l\lambda^{n-1-k-l}}{\frac{1}{(\lambda-\lambda_s)^{\nu_s}}\prod_{l=0}^{m-1}(\lambda-\lambda_l)^{\nu_l}}\,e^{\lambda(x-a)}\right\}_{\lambda=\lambda_s}. \tag{2.10}$$

Differentiation in this equality is conducted with respect to λ and into final result there is introduced $\lambda=\lambda_s$.

If all $\nu_s = 1$ $(s = 0,\ldots,m-1)$, then $m = n$ and

$$\frac{1}{\lambda-\lambda_s}\prod_{l=0}^{n-1}(\lambda-\lambda_l)\Big|_{\lambda=\lambda_s} = F^{(1)}(\lambda_s) = \sum_{i=0}^{n-1} p_i(n-i)\,\lambda_s^{n-i-1},$$

formulas (2.7) and (2.10) coincide.

At k = n-1 from equality (2.10) we find

$$Y_{n-1}(x) = \sum_{s=0}^{m-1}\frac{1}{(\nu_s-1)!}\,\frac{\partial^{\nu_s-1}}{\partial\lambda^{\nu_s-1}}\left\{\frac{e^{\lambda(x-a)}}{\frac{1}{(\lambda-\lambda_s)^{\nu_s}}\prod_{l=0}^{m-1}(\lambda-\lambda_l)^{\nu_l}}\right\}_{\lambda=\lambda_s}. \tag{2.11}$$

In the future there will be given examples of use of formulas (2.7) and (2.10).

We note that in the formula for $Y_k(x)$ $(k=0,...,n-1)$ the value of the coefficient p_n. does not enter. Its magnitude exerts an influence only on the value of the roots of the characteristic polynomial.

3. Recurrent Relationships Between Normal Fundamental Functions

From formulas (2.7) and (2.10) it is possible to establish the following basic relationship:

$$\frac{d}{dx}Y_k(x)=Y_{k-1}(x)-p_{n-k}Y_{n-1}(x) \qquad (k=0,\ 1,\ldots,\ n-1). \tag{3.1}$$

In this formula it is necessary to assume $Y_k(x)\equiv 0$ at $k<0$. Thus, for example, for derivative $Y_0(x)$ we shall have

$$\frac{d}{dx}Y_0(x)=-p_nY_{n-1}(x).$$

Equality (3.1) makes it possible to seek a system of normal fundamental functions of an equation with constant coefficients, if there is known, for example, function $Y_{n-1}(x)$. This method frequently is found to be in practical problems very effective, since the determination of $Y_{n-1}(x)$ by formulas (2.8) and (2.11) are relatively simple.

On the basis of equality (3.1)

$$\begin{aligned} Y_{n-2}(x)&=\frac{d}{dx}Y_{n-1}(x)+p_1Y_{n-1}(x),\\ Y_{n-3}(x)&=\frac{d}{dx}Y_{n-2}(x)+p_2Y_{n-1}(x), \end{aligned} \tag{3.2}$$

what results in a subsequent determination of all $Y_k(x)$ $(k=0,1,\ldots,n-1)$.

The recurrent relationship (3.1) makes it possible to express derivatives of the function $Y_k^{(l)}(x)$ $(l, k=0,\ldots,n-1)$ by a linear combination of the normal fundamental functions.

4. Solution of Inhomogeneous Equation

Suppose there is given the nonhomogeneous equation

$$\sum_{i=0}^{n} p_i y^{(n-i)}(x) = f(x) \qquad [p_0 = 1,\ y^{(0)}(x) = y(x)]. \tag{4.1}$$

The particular solution of equation (4.1) satisfying the zero initial conditions may be, as known, presented in the following form:

$$Y_*(x) = \int_a^x Y_{n-1}(x-s+a) f(s)\, ds, \tag{4.2}$$

which readily is verified by a direct substitution. This result can be established also by means of theory of integral residue, if equalities (2.8) and (2.11) are used.

For an explanation of the writing in the form (4.2) we shall present an illustrative example.

For the equation $\qquad y^{(2)}(x) + y(x) = f(x)$

we have

$$Y_0(x) = \cos(x-a),$$
$$Y_1(x) = \sin(x-a)$$
$$Y_*(x) = \int_a^x Y_{n-1}(x-s+a) f(s)\, ds = \int_a^x \sin(x-s) f(s)\, ds.$$

General solution of equation (4.1) will be thus:

$$y(x) = \sum_{k=0}^{n-1} y^{(k)}(a)\, Y_k(x) + \int_a^x Y_{n-1}(x-s+a) f(s)\, ds. \tag{4.3}$$

The solution in matrix form has the form (1.7).

5. Discontinuous Solutions

We shall seek the solution of equation (4.1), satisfying the given initial conditions and having given discontinuities (discontinuities of first order) of derivatives up to n--1 order inclusively.

Such a type of problem is encountered during calculation of concentrated influences. We note that coefficients of equation do not have discontinuities in entire interval of change x*.

Suppose the function $y^{(\nu)}(x)$ $(\nu=0, \ldots, n-1)$ has m discontinuities, located in the section $x=a_{\nu j}$ $(1 \leqslant j \leqslant m_\nu)$. The discontinuity of $y^{(\nu)}(x)$ in section $x=a_{\nu j}$, will be designated as:

$$y^{(\nu)}(a_{\nu j}+0)-y^{(\nu)}(a_{\nu j}-0)=\Delta_j^{(\nu)} \quad (\nu=0, 1, \ldots, n-1;\ j=1, \ldots, m_\nu), \tag{5.1}$$

Function $y^{(\nu)}(x)$, possessing given discontinuities must have the following structure:

$$y^{(\nu)}(x)=\varphi_\nu(x)+\sum_{j=1}^{m_\nu} S(x, a_{\nu j})\,\Delta_j^{(\nu)} \quad (\nu=0, 1, \ldots, n-1), \tag{5.2}$$

where $S(x, a_{\nu j})$ is a single function, determinated by the equality

$$S(x, a_{\nu j})=\begin{cases}0 & x<a_{\nu j},\\ 1 & x \geqslant a_{\nu j},\end{cases} \tag{5.3}$$

$\varphi_\nu(x)$ is a continuous function.

Function $y(x)$ which is a solution of the stated problem has to satisfy equation (4.1), the initial conditions and condition of discontinuities (5.2).

---

*Presence of discontinuities of function $y(x)$ and its first n--1 derivatives is not associated generally speaking, with the continuity of coefficients of the differential equation. There may take place also the reverse case, i.e., continuous solution in the presence of discontinuous coefficients.

It can be established that

$$y(x)=\sum_{k=0}^{n-1} y^{(k)}(a)\, Y_k(x)+\sum_{k=0}^{n-1}\sum_{j=1}^{m_k} S(x,\ a_{kj})\, \Delta_j^{(k)} Y_{kj}(x)+Y_*(x) \qquad (5.4)$$

corresponds to the indicated conditions, if functions of $Y_{kj}(x)$ satisfy the homogeneous differential equation (4.1) and the relationship

$$Y_{kj}^{(\nu)}(a_{kj})=\begin{cases}1 & \nu=k,\\ 0 & \nu\neq k\end{cases} \qquad (5.5)$$

$$(\nu,\ k=0,\ 1,\ \ldots,\ n-1).$$

Functions of $Y_{kj}(x)$ should be continuous together with the n-1 derivatives.

From the preceding it is clear that function $Y_{kj}(x)$ is normal fundamental function with the initial section $x=a_{kj}$. It is sufficient to assume

$$Y_{kj}(x)=Y_k(x-a_{kj}), \qquad (5.6)$$

so that all the indicated conditions above were satisfied.

Thus,

$$y(x)=\sum_{k=0}^{n-1} y^{(k)}(a)\, Y_k(x)+\sum_{k=0}^{n-1}\sum_{j=1}^{m_k} S(x,\ a_{kj})\, \Delta_j^{(k)} Y_k(x-a_{kj})+ \qquad (5.7)$$

$$+\int_a^x Y_{n-1}(x-s+a)\, f(s)\, ds.$$

The solution (5.7) can be written in a more symmetric form, if initial values of the function are considered as the given discontinuities after assuming

$$y(a)=\Delta_0^{(0)},\ \ldots,\ y^{(n-1)}(a)=\Delta_0^{(n-1)}$$

and by assuming all $a_{k0}=a \quad (k=0,\ \ldots,\ n-1).$

Now, equality (5.4) we shall write out as:

$$y(x)=\sum_{k=0}^{n-1}\sum_{j=0}^{m_k} S(x,\ a_{kj})\, \Delta_j^{(k)} Y_k(x-a_{kj})+Y_*(x). \qquad (5.8)$$

The solution of type (5.8) long ago was used in structural mechanics for equations of particular forms. By another method and in another form formula (5.8) was established by Sh. E. Mikeladze*. However, in the reasoning of the author an

*Sh. E. Mikeladze, Certain Problems of Structural Mechanics, State Engineering Publishing House, Moscow, 1948.

error crept in: equality (5.8) is correct only for an equation with constant coefficients, since only in this case, functions $Y_k(x-a_{kj})$ satisfy the corresponding differential equation.

Let us turn to the solution in matrix form. For convenience in writing we assume that in section $x = a_j$ there is in a general case a discontinuity of all derivatives which characterizes the "column - discontinuity".

$$[\Delta_j] = \begin{bmatrix} \Delta_j^{(0)} \\ \Delta_j^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \Delta_j^{(n-1)} \end{bmatrix}. \tag{5.9}$$

Certain elements of this column, of course, may be equal to zero.

The discontinuous solution of equation (4.1) in matrix form will be:

$$[y(x)] = \sum_{j=0}^{m} S(x, a_j)\,[Y(x-a_j)]\,[\Delta_j] + [Y_*(x)], \tag{5.10}$$

where $\underline{m}$ is the number of sections, in which there are discontinuities.

The initial column-discontinuity is

$$[\Delta_0] = \begin{bmatrix} y(a) \\ y^{(1)}(a) \\ \cdot \\ \cdot \\ \cdot \\ y^{(n-1)}(a) \end{bmatrix}. \tag{5.11}$$

The matrix $[Y(x-a_j)]$ has the form (1.8), where functions $Y_k(x)$ are replaced by $Y_k(x-a_j)$. If all elements $[\Delta_j]$ are given, then the discontinuity will be called independent. Frequently, however, there are encountered problems, in which discontinuity of derivative $y^{(\nu)}(x)$ in the section $x=a_j$ depends on values of function y(x) and its derivatives in the same section

(in the presence of discontinuities we shall for definiteness consider the left-hand values of functions in section $x=a_j$; the results almost without change are applicable for the case, when one should consider right side values).

Thus,

$$[\Delta_j] = [C_j]\,[y(a_j)]. \tag{5.12}$$

In this case the discontinuity will be called dependent, and $[C_j]$ is the matrix of dependent discontinuity. In theory of rods frequently there is used the equation of fourth order, in which

$$\begin{aligned} \Delta_j^{(2)} &= C_{20j}\,y(a_j) + C_{21j}\,y^{(1)}(a_j), \\ \Delta_j^{(3)} &= C_{30j}\,y(a_j) + C_{31j}\,y^{(1)}(a_j). \end{aligned} \tag{5.13}$$

Then

$$[C_j] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{20j} & C_{21j} & 0 & 0 \\ C_{30j} & C_{31j} & 0 & 0 \end{bmatrix}. \tag{5.14}$$

Discontinuous solutions can be applied both in case of concentrated influences on a system, and in the presence of discontinuity-like variation of the parameters in different sectors of the system, being described by one and the same differential equation (for example, flexure of stepped rod).

6. Examples of Applying Normal Fundamental Functions

As an example we shall consider the problem on flexure of a rod of constant section under action of given external forces (Fig. 1).

The differential equation has the form

$$y^{(4)}(x) = \frac{f(x)}{EJ}, \tag{6.1}$$

where y(x) is the sag of axis of rod; EJ--the strength of the section to flexure, f(x)--the distributed load per unit of length of beam.

Characteristic polynomial of the homogeneous equation

$$F(\lambda) = \lambda^4$$

has root $\lambda_0 = 0$ of fourth multiplicity $\nu_0 = 4$.

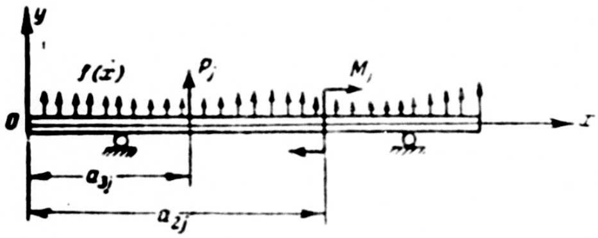

Fig. 1. Flexure of Rod.

From formula (2.10) we find (n = 4, m = 1, a = 0)

$$Y_3(x) = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} \{e^{\lambda x}\}_{\lambda=0} = \frac{x^3}{3!}.$$

From equalities (3.2) we determine

$$Y_2(x) = \frac{d}{dx} Y_3(x) = \frac{x^2}{2!},$$

$$Y_1(x) = \frac{d}{dx} Y_2(x) = \frac{x}{1!},$$

$$Y_0(x) = \frac{d}{dx} Y_1(x) = 1.$$

Of course, this system could have been written by not resorting to total results.

The function

$$Y_*(x) = \int_0^x \frac{(x-s)^3}{3!} f(s)\, ds.$$

At points of applying concentrated bending moments M_j and forces P_j there are discontinuities of the derivatives

$$\Delta_j^{(2)} = \frac{M_j}{EJ},$$

$$\Delta_j^{(3)} = \frac{P_j}{EJ}.$$

On the basis of equality (5.7) we obtain the known equation of the elastic line of a rod:

$$y(x)=y(0)+y^{(1)}(0)\,x+y^{(2)}(0)\frac{x^2}{2!}+y^{(3)}(a)\frac{x^3}{3!}+$$
$$+\sum_{j=1}^{m_0} S(x,\,a_{0j})\,\Delta_j^{(0)}+\sum_{j=1}^{m_1} S(x,\,a_{1j})\,\Delta_j^{(1)}(x-a_{1j})+$$
$$+\sum_{j=1}^{m_2} S(x,\,a_{2j})\frac{M_j}{EJ}\frac{(x-a_{2j})^2}{2!}+\sum_{j=1}^{m_3} S(x,\,a_{3j})\frac{P_j}{EJ}\frac{(x-a_{3j})^3}{3!}+$$
$$+\int_0^x \frac{(x-s)^3}{3!}\,f(s)\,ds. \tag{6.2}$$

In composing the equation it was assumed that the elastic line has in the sections a_{0j} discontinuities of sags, and in the sections a_{1j} --discontinuities of angles of rotation.

In practice, such a case can be encountered for a compound rod.

Equation (6.2) will be valid also for a rod with a graduated change of the section, if one were to introduce in corresponding sections the discontinuities

$$\Delta_j^{(2)}=M(b_j)\left[\frac{1}{EJ(b_j+0)}-\frac{1}{EJ(b_j-0)}\right],$$
$$\Delta_j^{(3)}=Q(b_j)\left[\frac{1}{EJ(b_j+0)}-\frac{1}{EJ(b_j-0)}\right].$$

where $M(b_j)$ and $Q(b_j)$ are the bending moment and the transverse force in the section $x=b_j$.

The normal fundamental matrix of equation (6.1) will be thus:

$$[Y(x)]=\begin{bmatrix} 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} \\ 0 & 1 & x & \frac{x^2}{2!} \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.3}$$

We consider now flexure vibrations of a weightless rod of constant section, carrying masses m_j and moments of inertia I_j (Fig. 2). In these sections there will be the dependent discontinuities.

$$\Delta_j^{(2)} = \frac{M_j}{EJ} = -\omega^2 \frac{I_j}{EJ} y^{(1)}(a_j),$$
$$\Delta_j^{(3)} = \frac{P_j}{EJ} = \omega^2 \frac{m_j}{EJ} y(a_j). \qquad (6.4)$$

where ω is the angular frequency of natural oscillations.

The column of the dependent discontinuity is expressed as:

$$\begin{bmatrix} \Delta_j^{(0)} \\ \Delta_j^{(1)} \\ \Delta_j^{(2)} \\ \Delta_j^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\omega^2 I_j}{EJ} & 0 & 0 \\ \frac{\omega^2 m_j}{EJ} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(a_j) \\ y^{(1)}(a_j) \\ y^{(2)}(a_j) \\ y^{(3)}(a_j) \end{bmatrix}. \qquad (6.5)$$

In sections, where supports are located there will be a discontinuity of the second and third derivative in accordance with magnitudes of reactive moment and force:

$$\Delta_j^{(2)} = K_{20j} y(a_j) + K_{21j} y^{(1)}(a_j),$$
$$\Delta_j^{(3)} = K_{30j} y(a_j) + K_{31j} y^{(1)}(a_j).$$

In majority of real cases $K_{20j}=0$ and $K_{31j}=0$; for elastic supports $K_{30j}<0$, $K_{21j}>0$.

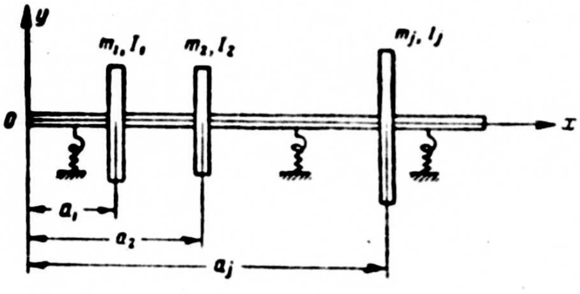

Fig. 2. Critical Speed of Shaft

The matrix of discontinuity for an elastic support has the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K_{20j} & K_{21j} & 0 & 0 \\ K_{30j} & K_{31j} & 0 & 0 \end{bmatrix}.$$

A rigid support may be taken into account by selecting the corresponding coefficients of rigidity of the elastic support. We note also that a rigid support which eliminates the section of linear mobility is equivalent to an application of $m_j \to \infty$; a support eliminating angular turns corresponds to an application $I_j \to \infty$. The solution of the problem in matrix form is expressed by the equality

$$|y(x)| = \sum_{j=0}^{m} S(x, a_j) [Y(x-a_j)] [C_j] [y(a_j)], \qquad (6.6)$$

where $[Y(x)]$ is the fundamental matrix of equation of flexure of rod; $[C_j]$ is matrix of discontinuity, corresponding to concentrated mass or elastic support.

Equation (6.6) is useful also for calculating for critical speed of a weightless shaft, loaded with separate disks.

For the case of forward synchronous precession* the matrix of discontinuity in section, where disk is located, will be thus:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\omega^2 I_j}{EJ} & 0 & 0 \\ \frac{\omega^2 m_j}{EJ} & 0 & 0 & 0 \end{bmatrix}.$$

where m_j is the mass of disk; I_j —is the equatorial moment of inertia. Thus, for example, the discontinuity of second derivative**

$$\Delta_j^{(2)} = \omega^2 \frac{I_j}{EJ} y^{(1)}(a_j).$$

---

*The concept of forward and reverse synchronous precessions is given in Chapter 4, Section 3.

**This equality in work of A. N. Krylov, "On Determining the Critical Speeds of a Revolving Shaft" (Academy of Sciences of USSR, Moscow, 1931) and later also in the work of Sh. E. Mikeladze "New Methods of Integrating Differential Equations" Moscow, State Theoretical Technical Publ. House 1951, is erroneously used with the minus sign. It follows from this, that the calculation of gyroscopic moment of disk during a forward synchronous precession does not increase, but lowers critical speed of shaft; this is not true.

In calculating a reverse synchronous precession for thin disks one should consider

$$\Delta_j^{(2)} = -\omega^2 \frac{3I_j}{EJ} y^{(1)}(a_j).$$

The calculation by equation (6.6) must be made in sequential order, in determining

$$[y(a_\nu)] = \sum_{j=0}^{\nu-1} [Y(a_\nu - a_j)] [C_j] [y(a_j)] \quad (\nu = 1, 2, 3, \ldots, m). \tag{6.7}$$

Suppose there is a certain number of unknown initial parameters (in considered problem there are two); then the same number of homogeneous boundary conditions should exist at x = b. As a result of the calculation we obtain

$$[\tilde{y}(b)] = [A_{ik}][\tilde{y}(a)]. \tag{6.8}$$

In this equality, the column $[\tilde{y}(a)]$ contains only the unknown initial parameters, and the column $[\tilde{y}(b)]$ -- those values of $y^{(\nu)}(b)$ $(\nu = 0, \ldots, n-1)$, which enter into boundary conditions at x = b.

In considered case boundary conditions are such:

$$[\tilde{y}(b)] = 0,$$

and then from relationships (6.8) there ensues

$$\det [A_{ik}] = 0. \tag{6.9}$$

Since the coefficients A_{ik} contain ω^2, then equality (6.9) is the characteristic equation for determining the eigen values.

Problem of calculating consists, essentially, in deteriming of elements of the matrix $[A_{ik}]$.

We discuss now in reference to the considered scheme of calculation one method, belonging A. N. Krylov (See "On Determining the Critical Speeds of Revolving Shaft"); this method makes it possible in most cases to simplify the calculation considerably.

Suppose the element, standing in the i-th line of column of initial parameters is unknown (for definiteness, we assume $i=2$).

Then we introduce the unit column

$$[y(a)^{*}] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{6.10}$$

where all elements -- zeros, except standing in line $\underline{i}$. With the initial column of (6.10) we conduct entire calculation, which determines elements of matrix $[A_{ik}]$, standing in column $\underline{i}$. After having made so many such calculations of so many unknown initial parameters, we determine all the elements of matrix.

We note that in a majority cases with use of equalities (6.7) there is no necessity to calculate all elements of matrices $[Y(a_{\cdot} - a_j)]$, which also facilitates the conduct of calculation.

In calculating the mass proper of rod (or shaft) we proceed from the equation

$$y^{(4)}(x) - \omega^2\alpha^4 y(x) = 0, \tag{6.11}$$

where y(x) -- amplitude sag of axis of rod;

$$\alpha^4 = \frac{\rho F}{EJ}$$

(here ρF – is mass of unit of length of shaft).

Characteristic polynomial of equation (6.11)

$$F(\lambda) = \lambda^4 - \omega^2\alpha^4$$

has the roots

$$\lambda_0 = \alpha\sqrt{\omega},\quad \lambda_1 = -\alpha\sqrt{\omega},\quad \lambda_2 = i\alpha\sqrt{\omega},\quad \lambda_3 = -i\alpha\sqrt{\omega}.$$

On the basis of formula (2.7) we obtain functions, introduced by A. N. Krylov:

$$\left.\begin{aligned}
Y_0(x) &= \sum_{s=0}^{3} \frac{\sum_{l=0}^{3} p_l\lambda_s^{3-l}}{4\lambda_s^3} e^{\lambda_s x} = \sum_{s=0}^{3} \frac{1}{4} e^{\lambda_s x} = \\
&= \frac{1}{2}(\cosh\alpha\sqrt{\omega}x + \cos\alpha\sqrt{\omega}x); \\
Y_1(x) &= \sum_{s=0}^{3} \frac{1}{4}\frac{e^{\lambda_s x}}{\lambda_s} = \frac{1}{2\alpha\sqrt{\omega}}(\sinh\alpha\sqrt{\omega}x + \sin\alpha\sqrt{\omega}x); \\
Y_2(x) &= \sum_{s=0}^{3} \frac{1}{4}\frac{e^{\lambda_s x}}{\lambda_s^2} = \frac{1}{2\alpha^2\omega}(\cosh\alpha\sqrt{\omega}x - \cos\alpha\sqrt{\omega}x); \\
Y_3(x) &= \sum_{s=0}^{3} \frac{1}{4}\frac{e^{\lambda_s x}}{\lambda_s^3} = \frac{1}{2\alpha^3\omega\sqrt{\omega}}(\sinh\alpha\sqrt{\omega}x - \sin\alpha\sqrt{\omega}x).
\end{aligned}\right| \tag{6.12}$$

The functions $Y_0(x), Y_1(x), Y_2(x)$ also can be obtained by means of recursion formulas (3.1). In using relationship (3.1), we will find fundamental matrix of equation (6.11)

$$[Y(x)] = \begin{bmatrix} Y_0(x) & Y_1(x) & Y_2(x) & Y_3(x) \\ \omega^2\alpha^4 Y_3(x) & Y_0(x) & Y_1(x) & Y_2(x) \\ \omega^2\alpha^4 Y_2(x) & \omega^2\alpha^4 Y_3(x) & Y_0(x) & Y_1(x) \\ \omega^2\alpha^4 Y_1(x) & \omega^2\alpha^4 Y_2(x) & \omega^2\alpha^4 Y_3(x) & Y_0(x) \end{bmatrix}. \qquad (6.13)$$

All preceding results relative to vibrations of rod remain in force, if only instead of matrix (6.3) we use matrix (6.13)*. To the same degree this refers also to calculating the critical numbers of revolution.

We present an example**, referring to determination of critical speeds of shaft with one disk (Fig. 3) in calculating the mass proper of shaft.

*If the mass of rod becomes vanishingly small $(\alpha \to 0)$, then matrix (6.13) transforms into matrix (6.3).

**This example is found in work of A. N. Krylov "On Determining the Critical Speeds of Revolving Shaft", in Sect. 10. However, in the solution errors crept in and made the result incorrect. Besides, the already mentioned error with the sign of the gyroscopic moment, in equation (58) of this work and subsequent equations, the expression for $y^{(1)}$ is written without consideration of influence of discontinuity.

The column of initial values has form

$$[\Delta_0]=[C_0][y(a)]=\begin{bmatrix}0\\ y^{(1)}(0)\\ 0\\ y^{(3)}(0)\end{bmatrix},$$

The column of first discontinuity ($x = a_1$)

$$[\Delta_1]=[C_1][y(a_1)]=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&\frac{\omega^2 I_1}{EJ}&0&0\\ \frac{\omega^2 m_1}{EJ}&0&0&0\end{bmatrix}\begin{bmatrix}y(a_1)\\ y^{(1)}(a_1)\\ y^{(2)}(a_1)\\ y^{(3)}(a_1)\end{bmatrix}=$$

$$=\begin{bmatrix}0\\0\\ \frac{\omega^2 I_1}{EJ}y^{(1)}(a_1)\\ \frac{\omega^2 m_1}{EJ}y(a_1)\end{bmatrix}. \qquad (6.14)$$

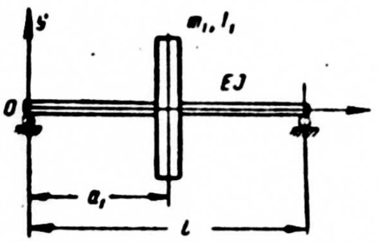

Fig. 3. Shaft with one disk.

Since boundary conditions at x = 1

$$y(l)=0, \quad y^{(2)}(l)=0,$$

then there must be calculated in matrixes of equation (6.7) only these two lines, and for determining the discontinuities, also $y(a_1)$ and $y^{(1)}(a_1)$. We shall have from relationships (6.7) and (6.13)

$$\begin{bmatrix}\boxed{y(a_1)}\\ \boxed{y^{(1)}(a_1)}\\ y^{(2)}(a_1)\\ y^{(3)}(a_1)\end{bmatrix}=\begin{bmatrix}\times & Y_1(a_1) & \times & Y_3(a_1)\\ \times & Y_0(a_1) & \times & Y_2(a_1)\\ \times&\times&\times&\times\\ \times&\times&\times&\times\end{bmatrix}\begin{bmatrix}0\\ y^{(1)}(0)\\ 0\\ y^{(3)}(0)\end{bmatrix}. \qquad (6.15)$$

Here, elements of matrix, not participating in calculations are marked $\times$.

From (6.15) it is evident

$$y(a_1)=y^{(1)}(0)\,Y_1(a_1)+y^{(3)}(0)\,Y_3(a_1),$$
$$y^{(1)}(a_1)=y^{(1)}(0)\,Y_0(a_1)+y^{(3)}(0)\,Y_2(a_1). \qquad (6.16)$$

For the section $\qquad a_2=l$

$$\begin{bmatrix} \boxed{y(l)} \\ y^{(1)}(l) \\ \boxed{y^{(2)}(l)} \\ y^{(3)}(l) \end{bmatrix} = \begin{bmatrix} \times & Y_1(l) & \times & Y_3(l) \\ \times & \times & \times & \times \\ \times & \omega^2\alpha^4 Y_3(l) & \times & Y_1(l) \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 0 \\ y^{(1)}(0) \\ 0 \\ y^{(3)}(0) \end{bmatrix} +$$

$$+\begin{bmatrix} \times & \times & Y_2(l-a_1) & Y_3(l-a_1) \\ \times & \times & \times & \times \\ \times & \times & Y_0(l-a_1) & Y_1(l-a_1) \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{\omega^2 I_1}{EJ} y^{(1)}(a_1) \\ \frac{\omega^2 m_1}{EJ} y(a_1) \end{bmatrix}.$$

Hence

$$\left.\begin{aligned} y(l) &= y^{(1)}(0)\, Y_1(l) + y^{(3)}(0)\, Y_3(l) + \frac{\omega^2 I_1}{EJ} Y_2(l-a_1)\times \\ &\times y^{(1)}(a_1) + \frac{\omega^2 m_1}{EJ} Y_3(l-a_1)\, y(a_1), \\ y^{(1)}(l) &= y^{(1)}(0)\, \omega^2\alpha^4 Y_3(l) + y^{(3)}(0)\, Y_1(l) + \frac{\omega^2 I_1}{EJ}\times \\ &\times Y_0(l-a_1)\, y^{(1)}(a_1) + \frac{\omega^2 m_1}{EJ} Y_1(l-a_1)\, y(a_1). \end{aligned}\right\} \tag{6.17}$$

In introducing values of (6.16) and considering condition (6.14), we obtain a system of homogeneous equations

$$\left.\begin{aligned} & y^{(1)}(0)\Big\{Y_1(l) + \frac{\omega^2 I_1}{EJ} Y_2(l-a_1)\, Y_0(a_1) + \\ & + \frac{\omega^2 m_1}{EJ} Y_3(l-a_1)\, Y_1(a_1)\Big\} + y^{(3)}(0)\Big\{Y_3(l) + \frac{\omega^2 I_1}{EJ}\times \\ & \times Y_2(l-a_1)\, Y_2(a_1) + \frac{\omega^2 m_1}{EJ} Y_3(l-a_1)\, Y_3(a_1)\Big\} = 0, \\ & y^{(1)}(0)\Big\{\omega^2\alpha^4 Y_3(l) + \frac{\omega^2 I_1}{EJ} Y_0(l-a_1)\, Y_0(a_1) + \\ & + \frac{\omega^2 m_1}{EJ} Y_1(l-a_1)\, Y_1(a_1)\Big\} + y^{(3)}(0)\Big\{Y_1(l) + \frac{\omega^2 I_1}{EJ}\times \\ & \times Y_0(l-a_1)\, Y_2(a_1) + \frac{\omega^2 m_1}{EJ} Y_1(l-a_1)\, Y_3(a_1)\Big\} = 0, \end{aligned}\right\} \tag{6.18}$$

or in matrix writing

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y^{(1)}(0) \\ y^{(3)}(0) \end{bmatrix} = 0.$$

The equality to zero det $[A_{ik}]$ gives the characteristic equation for determining ω^2.

The roots of equation are most simply found graphically by means of, constructing the function

$$F(\omega^2)=A_{11}A_{22}-A_{12}A_{21} \tag{6.19}$$

and in approaching limits by means of a linear interpolation between the points $F>0$ and $F<0$. We note that in matrixes there are calculated elements, standing in columns, corresponding to lines in columns of the discontinuities. For all intermediate sections there are maintained only lines necessary for calculating the discontinuities; for the last section -- lines corresponding to boundary conditions.

With a large number of discontinuities the presented method becomes cumbersome, since all the time it is necessary to determine the values $[y(a_\nu)]$ by the initial parameters.

In this case it is expedient to use method of A. N. Krylov, the application of which, we shall illustrate in this same example. We introduce the column of initial values

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and then from equalities (6.15) we obtain

$$y(a_1)^* = Y_1(a_1),$$
$$y^{(1)}(a_1)^* = Y_0(a_1).$$

Then we shall have

$$\begin{bmatrix} \boxed{y(l)^*} \\ y^{(1)}(l)^* \\ \boxed{y^{(2)}(l)^*} \\ y^{(3)}(l)^* \end{bmatrix} = \begin{bmatrix} \times & Y_1(l) & \times & \times \\ \times & \times & \times & \times \\ \times & \omega^2\alpha^4 Y_3(l) & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} \times & \times & Y_2(l-a_1) & Y_3(l-a_1) \\ \times & \times & \times & \times \\ \times & \times & Y_0(l-a_1) & Y_1(l-a_1) \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega^2 I_1 \quad Y_0(a_1) \\ \frac{\omega^2 m_1}{EJ} Y_1(a_1) \end{bmatrix},$$

hence

$$y(l)^* = A_{11} = Y_1(l) + \frac{\omega^2 I_1}{EJ} Y_2(l-a_1) Y_0(a_1) +$$

$$+ \frac{\omega^2 m_1}{EJ} Y_3(l-a_1) Y_1(a_1),$$

$$y^{(2)}(e)^* = A_{31} = \omega^2\alpha^4 Y_3(l) + \frac{\omega^2 I_1}{EJ} Y_0(l-a_1) Y_0(a_1) +$$

$$+ \frac{\omega^2 m_1}{EJ} Y_1(l-a_1) Y_1(a_1).$$

Analogously we find A_{12} and A_{22} and then function (6.19).

In taking into consideration the mass and moment of inertia of sections of rod or during action of constant, longitudinal force, the differential equation of the problem will have the form

$$y^{(4)}(x) + p_2 y^{(2)}(x) + p_4 y(x) = 0. \tag{6.20}$$

We shall determine the normal fundamental functions of this equation, by using results of Sec. 2 and Sec. 3.

The roots of characteristic polynomial

$$\lambda_{0,1} = \pm\sqrt{-\frac{p_2}{2} + \sqrt{\frac{p_2^2}{4} - p_4}},$$

$$\lambda_{2,3} = \pm\sqrt{-\frac{p_2}{2} - \sqrt{\frac{p_2^2}{4} - p_4}}$$

we designate

$$\lambda_0 = \mu, \quad \lambda_2 = \nu,$$
$$\lambda_1 = -\mu, \quad \lambda_3 = -\nu.$$

Further one should consider the relationship

$$\mu^2 + \nu^2 = -p_2,$$
$$\mu^2\nu^2 = p_4.$$

From equality (2.8) we obtain

$$Y_3(x) = \sum_{s=0}^{3} \frac{e^{\lambda_s x}}{4\lambda_s^3 + 2p_2\lambda_s} = \frac{e^{\mu x}}{4\mu^3 + 2p_2\mu} - \frac{e^{-\mu x}}{4\mu^3 + 2p_2\mu} +$$
$$+ \frac{e^{\nu x}}{4\nu^3 + 2p_2\nu} - \frac{e^{-\nu x}}{4\nu^3 + 2p_2\nu} = \frac{1}{\mu^2 - \nu^2}\left[\frac{1}{\mu}\sinh \mu x - \frac{1}{\nu}\sinh \nu x\right].$$

In using, now relationship (3.1), we find

$$Y_2(x) = \frac{d}{dx} Y_3(x) = \frac{1}{\mu^2 - \nu^2}[\cosh \mu x - \cosh \nu x],$$
$$Y_1(x) = \frac{d}{dx} Y_2(x) + p_2 Y_3(x) = -\frac{1}{\mu^2 - \nu^2}\left[\frac{\nu^2}{\mu}\sinh \mu x - \frac{\mu^2}{\nu}\sinh \nu x\right],$$
$$Y_0(x) = \frac{d}{dx} Y_1(x) = -\frac{1}{\mu^2 - \nu^2}[\nu^2 \cosh \mu x - \mu^2 \cosh \nu x].$$

Equality (3.1) makes it possible to determine all elements of fundamental matrix of equation (6.20).

7. Other Applications of Normal Fundamental Functions

Let us consider an axially symmetric deformation of a closed cylinderic shell (Fig. 4). The radial displacement of points of middle surface we designate as y(x).

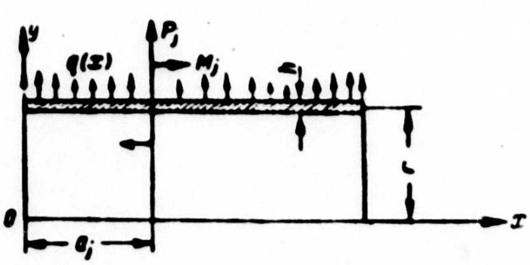

Fig. 4. Axially symmetric deformation of a cylindrical shell.

The differential equation for the function y(x) has the form

$$y^{(4)}(x)+4\beta^4 y(x)=f(x). \qquad (7.1)$$

In this equality

$$4\beta^4=\frac{Eh}{r^2D},$$

$$f(x)=\frac{1}{D}\left[q+\frac{Eh}{r}\alpha t_0-D(1+\mu)\frac{d^2}{dx^2}\left(\frac{\alpha\Delta t(x)}{h}\right)\right],$$

where $D=\frac{Eh^3}{12(1-\mu^2)}$ — is the cylindrical rigidity;

α — is the coefficient of linear expansion;

$t_0(x)$ — temperature of middle surface of shell;

$\Delta t(x)$ — drop in temperatures between outer and inner surfaces of shell.

Equation (7.1) conforms with the equation of flexure of beam on an elastic foundation, for which for the first time there were widely used by A. N. Krylov*, normal fundamental functions. We shall obtain an expression for these functions, using the general formula (2.7).

The characteristic polynomial

$$F(\lambda)=\lambda^4+4\beta^4$$

has the roots

$$\lambda_1=\beta(1+i);\quad \lambda_2=\beta(1-i);\quad \lambda_3=-\beta(1-i);\quad \lambda_4=-\beta(1+i).$$

---

*A. N. Krylov, On the Calcuation of Beams Lying on an Elastic Foundation. Academy of Sciences of USSR, Moscow, 1931.

Furthermore,

$$Y_0(x)=\sum_{s=0}^{3}\frac{\sum\limits_{l=0}^{3}p_l\lambda_s^{3-l}}{\sum\limits_{l=0}^{3}p_l(4-l)\lambda_s^{3-l}}e^{\lambda_s x}=\sum_{s=0}^{3}\frac{1}{4}e^{\lambda_s x}=$$

$$=\frac{1}{4}\left[e^{\beta(1+i)x}+e^{\beta(1-i)x}+e^{-\beta(1-i)x}+e^{-\beta(1+i)x}\right]=\cosh\beta x\cos\beta x;$$

$$Y_1(x)=\sum_{s=0}^{3}\frac{\sum\limits_{l=0}^{2}p_l\lambda_s^{2-l}}{4\lambda_s^3}e^{\lambda_s x}=\sum_{s=0}^{3}\frac{1}{4}\frac{e^{\lambda_s x}}{\lambda_s}=$$

$$=\frac{1}{4}\left[\frac{e^{\beta(1+i)x}}{\beta(1+i)}+\frac{e^{\beta(1-i)x}}{\beta(1-i)}-\frac{e^{-\beta(1-i)x}}{\beta(1-i)}-\frac{e^{-\beta(1+i)x}}{\beta(1+i)}\right]=$$

$$=\frac{1}{2\beta}(\cosh\beta x\sin\beta x+\sinh\beta x\cos\beta x); \qquad (7.2)$$

$$Y_2(x)=\sum_{s=0}^{3}\frac{\sum\limits_{l=0}^{1}p_l\lambda_s^{1-l}}{4\lambda_s^3}e^{\lambda_s x}=\sum_{s=0}^{3}\frac{1}{4}\frac{e^{\lambda_s x}}{\lambda_s^2}=$$

$$=\frac{1}{4\beta^2}\left[\frac{1}{2i}e^{\beta(1+i)x}-\frac{1}{2i}e^{\beta(1-i)x}-\frac{1}{2i}e^{-\beta(1-i)x}+\frac{1}{2i}e^{-\beta(1+i)x}\right]=$$

$$=\frac{1}{2\beta^2}\sinh\beta x\sin\beta x;$$

$$Y_3(x)=\sum_{s=0}^{3}\frac{\sum\limits_{l=0}^{0}p_l\lambda_s^{-l}}{4\lambda_s^3}=\sum_{s=0}^{3}\frac{1}{4}\frac{e^{\lambda_s x}}{\lambda_s^3}=$$

$$=\frac{1}{8\beta^3}\left[-\frac{e^{\beta(1+i)x}}{1+i}-\frac{e^{\beta(1-i)x}}{1+i}+\frac{e^{-\beta(1-i)x}}{1+i}+\frac{e^{-\beta(1+i)x}}{1-i}\right]=$$

$$=\frac{1}{4\beta^3}(\cosh\beta x\sin\beta x-\sinh\beta x\cos\beta x).$$

The normal fundamental matrix of equation (7.1) is determined on the basis of relationships (3.1). It has the form

$$[Y(x)]=\begin{bmatrix} Y_0(x) & Y_1(x) & Y_2(x) & Y_3(x)\\ -4\beta^4Y_3(x) & Y_0(x) & Y_1(x) & Y_2(x)\\ -4\beta^4Y_2(x) & -4\beta^4Y_3(x) & Y_0(x) & Y_1(x)\\ -4\beta^4Y_1(x) & -4\beta^4Y_2(x) & -4\beta^4Y_3(x) & Y_0(x)\end{bmatrix}. \qquad (7.3)$$

The particular solution of equation (7.1)

$$Y_*(x)=\int_a^x Y_3(x-s+a)\,f(s)\,ds. \tag{7.4}$$

If onto the shell in the section $x=a_j$ along a circular contour there are applied concentrated bending moments M_j and forces P_j, then in this section there exists a discontinuity of second and third derivative y(x)

$$[\Delta_j]=\begin{bmatrix}0\\0\\\frac{M_j}{D}\\\frac{P_j}{D}\end{bmatrix}.$$

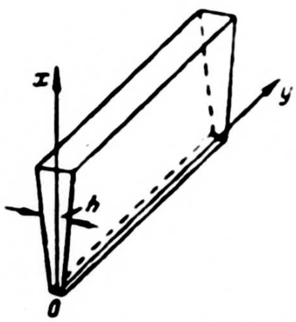

Fig. 5. Plane problem.

Solution of equation (7.1) is expressed by equality (5.10), where matrix $[Y(x)]$ is taken from relationship (7.3).

If shell in the section $x=a_j$ has reinforcing diaphragm (disk), then the discontinuity will be dependent, in which the matrix of the discontinuity will correspond to the matrix of elastic support.

The following example refers to one problem of theory of elasticity.

We shall consider the generalized problem on a plane strained state*. We assume that the external forces act in the middle of plane of a thin plate of variable thickness (Fig. 5).

Equations of equilibrium of element of plate will be such:

$$\begin{aligned}\frac{\partial(\sigma_x h)}{\partial x}+\frac{\partial(\tau_{xy} h)}{\partial y}+\rho hX=0,\\ \frac{\partial(\sigma_y h)}{\partial y}+\frac{\partial(\tau_{xy} h)}{\partial x}+\rho hY=0.\end{aligned} \tag{7.5}$$

*The more general statement of problem will be used later on.

In assuming that mass forces are absent and in introducing the function of stresses

$$\sigma_x = \frac{1}{h}\frac{\partial^2\Phi}{\partial y^2},$$

$$\sigma_y = \frac{1}{h}\frac{\partial^2\Phi}{\partial x^2},$$

$$\tau_{xy} = -\frac{1}{h}\frac{\partial^2\Phi}{\partial x\,\partial y},$$

we obtain from condition of continuity of the deformations

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{Eh}\left(\frac{\partial^2\Phi}{\partial x^2} - \mu\frac{\partial^2\Phi}{\partial y^2}\right) + \alpha t\right] + \frac{\partial^2}{\partial y^2}\left[\frac{1}{Eh}\left(\frac{\partial^2\Phi}{\partial y^2} - \mu\frac{\partial^2\Phi}{\partial x^2}\right) + \alpha t\right] + $$
$$+ 2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1+\mu}{Eh}\frac{\partial^2\Phi}{\partial x\,\partial y}\right) = 0. \qquad (7.6)$$

For a plate of constant thickness with constant parameters of elasticity and with an absence of heating we arrive at the well known biharmonic equation

$$\nabla^4\Phi(x, y) = 0. \qquad (7.7)$$

In investigating the state of strain in beams-walls the solution of equation (7.7) is sought in the form

$$\Phi(x, y) = \left.\begin{matrix}\sin \alpha y\\ \cos \alpha y\end{matrix}\right\}\psi(x).$$

In introducing the value $\Phi(x, y)$ into equality (7.7), we obtain the following differential equation for the function $\psi(x)$:

$$\psi^{(4)}(x) - 2\alpha^2\psi^{(2)}(x) + \alpha^4\psi(x) = 0. \qquad (7.8)$$

The characteristic polynomial has two roots of second multiplicity

$$\lambda_0 = \alpha, \quad \lambda_1 = -\alpha.$$

For determinating $\Psi_3(x)$ we shall use formula (2.11).

We have at a = 0

$$\Psi_3(x) = \sum_{s=0}^{1} \frac{\partial}{\partial\lambda}\left\{\frac{e^{\lambda x}}{\frac{1}{(\lambda-\lambda_s)^2}\prod\limits_{i=0}^{1}(\lambda-\lambda_i)^2}\right\}_{\lambda=\lambda_s} = \frac{\partial}{\partial_\lambda}\left\{\frac{e^{\lambda x}}{(\lambda-\lambda_1)^2}\right\}_{\lambda=\lambda_0} +$$

$$+\frac{\partial}{\partial\lambda}\left\{\frac{e^{\lambda x}}{(\lambda-\lambda_0)^2}\right\}_{\lambda=\lambda_1} = \frac{1}{2\alpha^3}(\alpha x\cosh\alpha x - \sinh\alpha x).$$

Further on basis of equalities (3.2) we find

$$\Psi_2(x) = \frac{1}{2\alpha^2}\alpha x\sinh\alpha x,$$

$$\Psi_1(x) = \frac{1}{2\alpha}(3\sinh\alpha x - \alpha x\cosh\alpha x),$$

$$\Psi_0(x) = \frac{1}{2}(2\cosh\alpha x - \alpha x\sinh\alpha x).$$

The application of normal fundamental functions introduces into the considered problem a number of simplifications in satisfying the boundary conditions. The use of the general method in Sec. 5 makes it possible to construct a solution in the presence of a discontinuity-like variation in the thickness of the plate, since for each of the sections, equation (7.8) remains in force.

8. Normal Fundamental Functions of Euler's Equation

We now consider the homogeneous Euler equation

$$y^{(n)}(x) + \frac{c_1}{x}y^{(n-1)}(x) + \ldots + \frac{c_n}{x^n}y(x) = 0 \tag{8.1}$$

or

$$\sum_{i=0}^{n}\frac{c_i}{x^i}y^{(n-i)}(x) = 0 \tag{8.2}$$

$$(c_0 = 1, \quad y^{(0)}(x) = y(x)),$$

where c_i— are constant coefficients.

By substitution

$$x=e^{\xi} \tag{8.3}$$

equation (8.2) reduces to an equation with constant coefficients:

$$\sum_{i=0}^{n} p_i y^{(n-i)}(\xi)=0 \tag{8.4}$$
$$\left(y^{(n-i)}(\xi)=\frac{d^{n-i}}{d\xi^{n-i}}y(\xi)\right).$$

The characteristic polynomial of equation (8.4)

$$F(\lambda)=\sum_{i=0}^{n} p_i\lambda^{n-i}. \tag{8.5}$$

The coefficients p_i are the simplest of all to determine by using the identity of characteristic polynomials of equations (8.4) and (8.2), if there is introduced into the latter

$$y(x)=x^{\lambda}.$$

In decomposing the characteristic polynomial of equation (8.2) by degrees of λ, we obtain the values of p_i.

Suppose there is given, for example, an equation of fourth order

$$y^{(4)}(x)+\frac{c_1}{x}y^{(3)}(x)+\frac{c_2}{x^2}y^{(2)}(x)+\frac{c_3}{x^3}y^{(1)}(x)+\frac{c_4}{x^4}y(x)=0.$$

The characteristic polynomial of this equation

$$F(\lambda)=\lambda(\lambda-1)(\lambda-2)(\lambda-3)+\lambda(\lambda-1)(\lambda-2)c_1+$$
$$+\lambda(\lambda-1)c_2+\lambda c_3+c_4.$$

After decomposition by degrees of λ we obtain

$$F(\lambda)=\lambda^4+\lambda^3(-6+c_1)+\lambda^2(11-3c_1+c_2)+$$
$$+\lambda(-6+2c_1-c_2+c_3)+c_4.$$

Thus, we find

$$\sum_{i=0}^{4} p_i y^{(n-i)}(\xi)=0,$$

$$p_0=1,$$
$$p_1=-6+c_1,$$
$$p_2=11-3c_1+c_2,$$
$$p_3=-6+2c_1-c_2+c_3,$$
$$p_4=c_4.$$

We shall consider the solution of equation (8.1) at the values $a < x < b$ $(a > 0$ since x = 0 is a singular point of the equation). Suppose the value x = a corresponds to $\xi = \alpha$, $\alpha = \ln a$.

For equation (8.4) on basis of indicated earlier formulas, there may be found the normal fundamental functions and then

$$y(\xi) = \sum_{k=0}^{n-1} y^{(k)}(\alpha) Y_k(\xi),$$

in which

$$\frac{d^i}{d\xi^i} Y_k(\alpha) = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases}$$

Then arises the question, will the functions $Y_k(\xi)$, if we assume in them $\xi = \ln x$, be the normal fundamental functions of equation (8.1).

It is possible to establish that this will take place for equations up to the second order inclusively and it is found invalid for equations of higher orders. However, in any case

$$\frac{d^k}{dx^k} Y_k(x) \Big|_{x=a} = \frac{1}{a^k},$$

and therefore even for second order equations functions $Y_0(x)$ and $aY_1(x)$. will form the normal fundamental system. If there is an inhomogeneous Euler equation

$$\sum_{i=0}^{n} \frac{c_i}{x^i} y^{(n-i)}(x) = f(x), \tag{8.6}$$

then by substitution (8.3),it reduces to the equation

$$\sum_{i=0}^{n} p_i y^{(n-i)}(\xi) = f(\xi), \tag{8.7}$$

general solution of which has the form

$$y(\xi) = \sum_{k=0}^{n-1} y^{(k)}(\alpha) Y_k(\xi) + \int_{\alpha}^{\xi} Y_{n-1}(\xi - s + \alpha) f(s)\, ds. \tag{8.8}$$

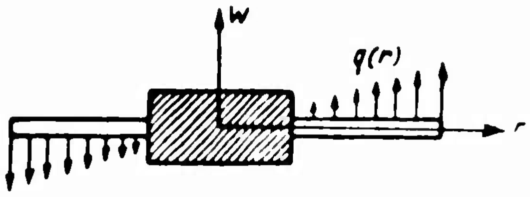

Fig. 6. Flexure of disk with a cosinusoidal load.

Let us consider the following example (Fig. 6). The differential equation of the flexure of a circular plate of constant thickness under action of cosinusoidal load has the form

$$\nabla^4 w = \frac{q(r)}{D}\cos\theta,$$

where

$$\nabla^4 w = \frac{\partial^4 w}{\partial r^4} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \theta^4} + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{2}{r^3}\frac{\partial^3 w}{\partial r \partial \theta} + \frac{4}{r^4}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r};$$

D -- cylindrical rigidity.

In assuming a solution in the form

$$w(r, \theta) = \varphi(r)\cos\theta,$$

we obtain for $\varphi(r)$ the Euler equation

$$\varphi^{(4)}(r) + \frac{2}{r}\varphi^{(3)}(r) - \frac{3}{r^2}\varphi^{(2)}(r) + \frac{3}{r^3}\varphi^{(1)}(r) - \frac{3}{r^4}\varphi(r) = \frac{q(r)}{D}.$$

By substitution

$$r = e^{\xi}$$

it reduces to following ordinary differential equation of fourth order:

$$\frac{d^4\varphi}{d\xi^4} - 4\frac{d^3\varphi}{d\xi^3} + 2\frac{d^2\varphi}{d\xi^2} + 4\frac{d\varphi}{d\xi} - 3\varphi = \frac{q(\xi)}{D}e^{4\xi}. \qquad (8.9)$$

The roots of characteristic polynomial

$$F(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 4\lambda - 3$$

are:

$$\lambda_0 = 1, \quad (\nu_0 = 2), \quad \lambda_1 = -1, \quad \lambda_2 = 3$$

(root λ_0 has second multiplicity).

From formula (2.11) we obtain

$$\Phi_3(\xi)=\frac{1}{1!}\frac{\partial}{\partial\lambda}\left\{\frac{e^{\lambda(\xi-a)}}{(\lambda+1)(\lambda-3)}\right\}_{\lambda=1}+\frac{e^{\lambda(\xi-a)}}{4\lambda^3-12\lambda^2+4\lambda+4}\bigg|_{\lambda=-1}+$$

$$+\frac{e^{\lambda(\xi-a)}}{4\lambda^3-12\lambda^2+4\lambda+4}\bigg|_{\lambda=3}=-\frac{1}{4}(\xi-a)e^{\xi-a}-\frac{1}{16}e^{-(\xi-a)}+\frac{1}{16}e^{3(\xi-a)}.$$

By means of equalities (3.2) we find

$$\left.\begin{aligned}\Phi_2(\xi)&=\frac{1}{4}e^{\xi-a}+\frac{3}{4}(\xi-a)e^{\xi-a}+\frac{5}{16}e^{-(\xi-a)}-\frac{1}{16}e^{3(\xi-a)},\\ \Phi_1(\xi)&=\frac{1}{2}e^{\xi-a}+\frac{1}{4}(\xi-a)e^{\xi-a}-\frac{7}{16}e^{-(\xi-a)}-\frac{1}{16}e^{3(\xi-a)},\\ \Phi_0(\xi)&=\frac{3}{4}e^{\xi-a}-\frac{3}{4}(\xi-a)e^{\xi-a}+\frac{3}{16}e^{-(\xi-a)}+\frac{1}{16}e^{3(\xi-a)}.\end{aligned}\right\} \qquad (8.10)$$

The particular solution

$$\Phi_*(\xi)=\int_a^{\xi}\Phi_3(\xi-s+a)\frac{q(s)}{D}e^{4s}ds. \qquad (8.11)$$

Thus,

$$\varphi(\xi)=\sum_{k=0}^{3}\varphi^{(k)}(a)\Phi_k(\xi)+\Phi_*(\xi).$$

Let us apply now the obtained solution for a plate (disk) of variable thickness. For this purpose we shall divide the plate into sectors of constant thickness with sections $\xi=a_j$ $(j=0,1,\ldots,)$. We shall designate the cylindrical rigidity

$$D(a_j-0)=D_{j-1},$$
$$D(a_j+0)=D_j$$

and correspondingly

$$\varphi(a_j-0)=\varphi_{j-1},$$
$$\varphi(a_j+0)=\varphi_j.$$

From condition of equality of the bending moments per unit of length of cylindrical section we obtain

$$D_j[\varphi_j^{(2)}-(1-\mu)\varphi_j^{(1)}-\mu\varphi_j]=D_{j-1}[\varphi_{j-1}^{(2)}-(1-\mu)\varphi_{j-1}^{(1)}-\mu\varphi_{j-1}]. \qquad (8.12)$$

Owing to the continuity of deformation

$$\varphi_j = \varphi_{j-1},$$
$$\varphi_j^{(1)} = \varphi_{j-1}^{(1)} \quad (\xi = a_j)$$

and then from (8.12) we obtain

$$\Delta_j^{(2)} = \varphi_j^{(2)} - \varphi_{j-1}^{(2)} = \left(\frac{D_{j-1}}{D_j} - 1\right)[\varphi_{j-1}^{(2)} - (1-\mu)\varphi_{j-1}^{(1)} - \mu\varphi_{j-1}].$$

From condition of equality of the total transverse force we find

$$D_j[\varphi_j^{(3)} - 2\varphi_j^{(2)} - (2-\mu)\varphi_j^{(1)} + (3-\mu)\varphi_j] =$$
$$= D_{j-1}[\varphi_{j-1}^{(3)} - 2\varphi_{j-1}^{(2)} - (2-\mu)\varphi_{j-1}^{(1)} + (3-\mu)\varphi_{j-1}].$$

It follows from this

$$\begin{aligned}\Delta_j^{(3)} = \varphi_j^{(3)} - \varphi_{j-1}^{(3)} = \left(\frac{D_{j-1}}{D_j} - 1\right)[\varphi_{j-1}^{(3)} - 2\varphi_{j-1}^{(2)} - (2-\mu)\varphi_{j-1}^{(1)} + \\ + (3-\mu)\varphi_{j-1}] + 2\Delta_j^{(2)} = \left(\frac{D_{j-1}}{D_j} - 1\right)\times \\ \times[\varphi_{j-1}^{(3)} - (4-3\mu)\varphi_{j-1}^{(1)} + 3(1-\mu)\varphi_{j-1}].\end{aligned} \tag{8.13}$$

Thus, we shall have

$$[\varphi(\xi)] = \sum_{j=0}^{m} S(x, a_j)[\Phi_k(\xi - a_j)][C_j][\varphi(a_j)] + [\Phi_*(\xi)]. \tag{8.14}$$

Here $[\Phi_k(\xi - a_j)]$ – is the normal fundamental matrix of the homogeneous equation (8.9). Its elements are determined from the relationship (3.1) or by direct differentiation of equalities (8.10).

The matrix of discontinuity has the form

$$[C_j] = \left(\frac{D_{j-1}}{D_j} - 1\right)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mu & -(1-\mu) & 1 & 0 \\ 3(1-\mu) & -(4-3\mu) & 0 & 1 \end{bmatrix}$$

In equality (8.14) into $[\varphi(a_j)]$ the left-hand values enter

$$[\varphi(a_j)] = [\varphi_{j-1}].$$

We shall dwell on determining the column of a particular solution. At $a_{j-1} < \xi < a_j$ from equality (8.11) we obtain

$$\Phi_*(\xi) = \sum_{\nu=1}^{j-1} \int_{a_{\nu-1}}^{a_\nu} \Phi_3(\xi - s + a) \frac{q(s)}{D_{\nu-1}} e^{\lambda s} ds + \int_{a_{j-1}}^{\xi} \Phi_3(\xi - s + a) \frac{q(s)}{D_{j-1}} e^{\lambda s} ds. \quad (8.15)$$

In determining the elements $[\Phi_*(\xi)]$ one should not forget that all components on right-hand side of the equality depend on ξ.

9. Application to the Integration of Equations with Variable Coefficients

Suppose there is a linear differential equation with the variable coefficients

$$\sum_{i=0}^{n} p_i(x) y^{(n-i)}(x) = f(x) \quad [p_0(x) = 1, \quad y^{(0)}(x) = y(x)] \quad (9.1)$$

and there is sought the solution of equation in the interval $a<x<b$. The mean values of coefficients can be determined for example, in the following manner:

$$\overset{*}{p}_{i\,cp} = \frac{1}{b-a} \int_a^b p_i(x)\, dx. \quad (9.2)$$

We write out equation (9.1) in the following form:

$$\sum_{i=0}^{n} p_{i\,cp} y^{(n-i)}(x) = f(x) + \sum_{i=1}^{n} (p_{i\,cp} - p_i(x)) y^{(n-i)}(x) \quad (p_{0cp} = 1, \quad y^{(0)}(x) = y(x)). \quad (9.3)$$

Suppose $\widetilde{Y}_k(x) \quad (k=0,\dots,n-1)$ — are the normal fundamental functions of equation with constant coefficients:

$$\sum_{i=0}^{n} p_{i\,cp} y^{(n-i)}(x) = 0 \quad (9.4)$$

(with the initial section x = a).

Solution of equation (9.3) with an arbitrary right-hand side is written out as:

$$y(x) = \sum_{k=0}^{n-1} y^{(k)}(a) \widetilde{Y}_k(x) + \widetilde{Y}_*(x) + \int_a^x \widetilde{Y}_{n-1}(x - s + a) \times \sum_{i=1}^{n} (p_{i\,cp} - p_i(s)) y^{(n-i)}(s)\, ds, \quad (9.5)$$

* cp equals average.

where

$$\tilde{Y}_{0}(x)=\int_{a}^{x} Y_{n-1}(x-s+a) f(s)\, ds. \tag{9.6}$$

In differentiating equality (9.5), we find

$$y^{(\nu)}(x)=\sum_{k=0}^{n-1} y^{(k)}(a)\, \tilde{Y}_{k}^{(\nu)}(x)+\tilde{Y}_{0}^{(\nu)}(x)+\int_{a}^{x}\tilde{Y}_{n-1}^{(\nu)}(x-s+a)\times$$
$$\times\left[\sum_{i=1}^{n}(p_{icp}-p_i(s))\, y^{(n-i)}(s)\, ds\right] \qquad (\nu=0, 1, \ldots, n-1). \tag{9.7}$$

By introducing, as previously, the column-solution

$$[y(x)]=\begin{bmatrix} y(x) \\ y^{(1)}(x) \\ \vdots \\ y^{(n-1)}(x) \end{bmatrix}, \tag{9.8}$$

we shall write out system (9.7) in following form:

$$[y(x)]=[\tilde{Y}(x)]\,[y(a)]+[\tilde{Y}_{0}(x)]+\int_{a}^{x}[K(x, s)]\,[y(s)]\, ds. \tag{9.9}$$

Equation (9.9) represents the matrix integral Volterra equation.

The matrix is the nucleus of the equation

$$[K(x, s)]=$$
$$=\begin{bmatrix} \tilde{Y}_{n-1}(x-s+a)(p_{ncp}-p_n(s))\ldots\tilde{Y}_{n-1}(x-s+a)(p_{1cp}-p_1(s)) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \tilde{Y}_{n-1}^{(n-1)}(x-s+a)(p_{ncp}-p_n(s))\ldots\tilde{Y}_{n-1}^{(n-1)}(x-s+a)(p_{1cp}-p_1(s)) \end{bmatrix}. \tag{9.10}$$

Equation (9.9) is solved by method of successive approximations, in which the process is convergent, if all coefficients $p_i(x)$ are limited in interval $a<x<b$. Furthermore, it is possible to show that solution of equation (9.9) results in determining the normal fundamental functions of the homogeneous equation (9.1) and the particular solution, satisfying the zero initial conditions.

Suppose it is necessary to determine the k-th normal fundamental function of equation (9.1) $Y_k(x)$.

We shall assume in the column of initial values all $y^{(\nu)}(a)$, $(\nu\neq k)$ are equal to zero, and $y^{(k)}(a)=1$. From equation (9.9) at $\tilde{Y}_{0}(x)=0$ and, consequently, $f(x)=0$, we obtain

$$[Y_k(x)] = \left[\widetilde{Y}_k(x)\right] + \int_a^x [K(x, s)]\,[Y_k(s)]\,ds. \tag{9.11}$$

The first approximation is

$$[Y_{k(1)}(x)] = \left[\widetilde{Y}_k(x)\right]; \tag{9.12}$$

second approximation is

$$[Y_{k(2)}(x)] = \left[\widetilde{Y}_k(x)\right] + \int_a^x [K(x, s)]\left[\widetilde{Y}_k(s)\right] ds \tag{9.13}$$

et cetera.

The presentation of solution in the form of series leads to the same results

$$[Y_k(x)] = \sum_{i=1}^{\infty} [\Phi_i(x)], \tag{9.14}$$

in which

$$[\Phi_1(x)] = [\widetilde{Y}_k(x)]. \tag{9.15}$$

Calculation is terminated, when the difference between two successive approximations can be assumed to be **negligibly** small or with the use of equalities (9.14), when considered terms of the series is small in comparison with the sum of preceding terms. In the process of successive approximations there is no necessity to calculate all the elements of the column $[Y_k(x)]$; only the elements, for which $p_{i\,cp} - p_i(s) \not\equiv 0$. will be subject to calculation. Others required for calculating the derivatives $Y_k(x)$, are computed from equation (9.11) after the indicated elements have been determined.

We shall consider as example the Bessel equation of order $\frac{1}{2}$:

$$y^{(2)}(x) + \frac{1}{x} y^{(1)}(x) + \left(1 - \frac{1}{4x^2}\right) y(x) = 0. \tag{9.16}$$

We shall seek an approximate solution of this equation in the interval $\left(\frac{\pi}{6}, \pi\right)$.

The mean values of coefficients by formula (9.2) are equal to

$$\begin{aligned} p_{1\,cp} &= \frac{6}{5\pi} \ln 6, \\ p_{2\,cp} &= 1 - \frac{3}{2\pi^2}. \end{aligned} \tag{9.17}$$

The normal fundamental functions of the equation with constant coefficients

$$y^{(2)}(x)+p_{1\,cp}y^{(1)}(x)+p_{2\,cp}y(x)=0 \qquad (9.18)$$

will be such:

$$\left.\begin{aligned}\tilde{Y}_0(x)&=\frac{1}{2\sqrt{\frac{p_{1\,cp}^2}{4}-p_{2\,cp}}}\left[\lambda_0 e^{\lambda_1(x-a)}-\lambda_1 e^{\lambda_0(x-a)}\right],\\ \tilde{Y}_1(x)&=-\frac{1}{2\sqrt{\frac{p_{1\,cp}^2}{4}-p_{2\,cp}}}\left[e^{\lambda_0(x-a)}-e^{\lambda_1(x-a)}\right],\end{aligned}\right\} \qquad (9.19)$$

where

$$\lambda_0=-\frac{p_{1cp}}{2}+\sqrt{\frac{p_{1cp}^2}{4}-p_{2cp}};$$

$$\lambda_1=-\frac{p_{1cp}}{2}-\sqrt{\frac{p_{1cp}^2}{4}-p_{2cp}}.$$

In the considered case $a=\frac{\pi}{6}$.

In considering (9.17), we find

$$\tilde{Y}_0(x)=\frac{1}{0,8549}e^{-0,3422\left(x-\frac{\pi}{6}\right)}\left[0,8549\cos 0,8549\left(x-\frac{\pi}{6}\right)+\right.$$

$$\left.+0,3422\sin 0,8549\left(x-\frac{\pi}{6}\right)\right],$$

$$\tilde{Y}_1(x)=\frac{1}{0,8549}e^{-0,3422\left(x-\frac{\pi}{6}\right)}\sin 0,8549\left(x-\frac{\pi}{6}\right).$$

We shall determine the normal fundamental function $Y_0(x)$ of equation (9.16). From relationship (9.11) we shall have

$$\begin{bmatrix}Y_0(x)\\ Y_0^{(1)}(x)\end{bmatrix}=\begin{bmatrix}\tilde{Y}_0(x)\\ \tilde{Y}_0^{(1)}(x)\end{bmatrix}+$$

$$+\int_0^x\begin{bmatrix}\tilde{Y}_1(x-s+a)\left(p_{1cp}-\frac{1}{s}\right), & \tilde{Y}_1(x-s+a)\left(p_{2cp}-1+\frac{1}{4s^2}\right)\\ \tilde{Y}_1^{(1)}(x-s+a)\left(p_{1cp}-\frac{1}{s}\right), & \tilde{Y}_1^{(1)}(x-s+a)\left(p_{2cp}-1+\frac{1}{4s^2}\right)\end{bmatrix}\times$$

$$\times\begin{bmatrix}Y_0(s)\\ Y_0^{(1)}(s)\end{bmatrix}ds.$$

Hence the first approximation for the function $Y_0(x)$

$$Y_{0(1)}(x) = \widetilde{Y}_0(x). \tag{9.20}$$

Second approximation

$$Y_{0(2)}(x) = \widetilde{Y}_0(x) + \int_0^x \widetilde{Y}_1(x-s+a) \times$$

$$\times\left[\left(p_{1cp} - \frac{1}{s}\right)\widetilde{Y}_0(s) + \left(p_{2cp} - 1 + \frac{1}{4s^2}\right)\widetilde{Y}_0^{(1)}(s)\right]ds. \tag{9.21}$$

In Table 1 the calculation of second approximation is explained. In the calculation the interval $\frac{\pi}{6}, \pi$ is divided into ten equal sectors. For compiling tables of values of the functions $\widetilde{Y}_1(x-s+a)$ is calculated as soon as first column.

Since s = a, then into the column the values $\widetilde{Y}_1(x)$. are entered. All other columns are filled in such a manner so that the elements on secondary diagonals are identical.

This ensues from the circumstance that in the division into equal sectors for these diagonals the magnitude $x-s+a$ remains constant.

In column 12 there is introduced an expression, standing in equality (9.21) in brackets. Furthermore each element in column 12 is multiplied by the corresponding value $\widetilde{Y}_1(x-s+a)$ and the result is summarized by the trapezoidal rule for all values in a given data line. This sum is contained in column 13. In column 14 there is given the value

$$Y_{0(1)}(x) = \widetilde{Y}_0(x),$$

then -- value of the magnitude $Y_{0(2)}(x)$, $Y_{0(3)}(x)$ and the accurate value $Y_0(x)$. In the given case, a third approximation gives an accuracy, adequate for a majority of engineering problems (Fig. 7).

Sometimes it is expedient to apply another variant of the presented method, which consists of the circumstance that equations with constant coefficients strive to obtain a form as simple as possible.

Table 1. Solution of Equation (9.16)

(a) Строка	(b) Столбец	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
		$\tilde{Y}_1(x-s+a)$											[ ]	$\int_a^x (\ )ds$	$Y_{0(1)}$	$Y_{0(2)}$	$Y_{0(3)}$	Y_0
	x / s	0,524	0,785	1,047	1,309	1,571	1,833	2,094	2,356	2,618	2,880	3,142						
1	0,524	0											0,760	0	1,000	1,000	1,000	1,000
2	0,785	0,237	0										0,365	0,024	0,973	0,997	0,997	0,990
3	1,047	0,423	0,237	0									0,165	0,065	0,899	0,964	0,959	0,950
4	1,309	0,556	0,423	0,237	0								0,033	0,106	0,789	0,895	0,885	0,874
5	1,571	0,638	0,556	0,423	0,237	0							−0,059	0,137	0,656	0,793	0,777	0,766
6	1,833	0,672	0,638	0,556	0,423	0,237	0						−0,119	0,152	0,509	0,661	0,642	0,632
7	2,094	0,666	0,672	0,638	0,556	0,423	0,237	0					−0,151	0,149	0,361	0,510	0,488	0,478
8	2,356	0,625	0,606	0,672	0,638	0,556	0,423	0,237	0				−0,161	0,129	0,216	0,345	0,322	0,313
9	2,618	0,557	0,625	0,666	0,672	0,638	0,556	0,423	0,237	0			−0,153	0,096	0,085	0,181	0,154	0,147
10	2,880	0,472	0,557	0,625	0,666	0,672	0,638	0,556	0,423	0,237	0		−0,138	0,053	−0,030	0,023	−0,006	−0,013
11	3,142	0,375	0,472	0,557	0,625	0,666	0,672	0,638	0,556	0,423	0,237	0	−0,132	0,006	−0,124	−0,118	−0,153	−0,158

KEY: (a) Line; (b) Column.

We shall turn to the generalized problem of a plane stress condition and shall assume that $h=h(x)$ (See Fig. 5). Then for $\psi(x)$ we obtain the equation

$$\psi^{(4)}(x)-2a^2\psi^{(2)}(x)+p_3(x)\psi^{(1)}(x)+a^4\psi(x)=0,$$

where

$$p_3(x)=2\frac{d}{dx}(Eh)\frac{1+\mu}{Eh}.$$

By transferring the term, containing $\psi^{(1)}(x)$, into the right-hand side of equality, we arrive at the equation

$$\psi(x)=\sum_{k=0}^{3}\psi^{(k)}(0)\widetilde{\Psi}_k(x)-\int_0^x\widetilde{\Psi}_3(x-s)\,p_3(s)\,\psi^{(1)}(s)\,ds,$$

where $\widetilde{\Psi}_k(x)$ — is the normal fundamental functions of equation (7.8).

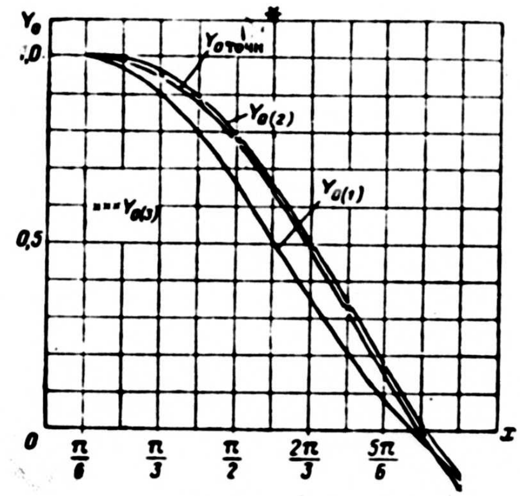

Fig. 7. The Comparison of accurate and approximate solutions

The first approximation in this problem expresses the stress condition in a plate of constant thickness.

In conclusion, we shall say several words on another method of using equations with constant coefficients for solving equation (9.1), which sometimes is applied in engineering problems. In this method, the total interval of variation of x is divided into sectors within limits of which the coefficients $p_i(x)$ are assumed constant. And here the use of the solution in form (1.7) gives an essential advantage.

Thus, determination of $[y(b)]$ reduces to the multiplication of matrices, corresponding to individual sectors.

* Yo accurate value.

CHAPTER 2

NORMAL FUNDAMENTAL FUNCTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

There are considered two methods of approximate determination of normal fundamental functions: method of successive approximations and method of linear approximation.

Both methods are used for solving the normal integral equation, to which a differential equation under initial Cauchy conditions reduces.

1. Statement of Problem

There is given a linear differential equation of n-th order with variable coefficients

$$y^{(n)}(x)+p_1(x)y^{(n-1)}(x)+\ldots+p_n(x)y(x)=f(x). \qquad (1.1)$$

It is required to find the solution of this equation in certain interval of variation of x($a \leqslant x \leqslant b$).

The functions $p_i(x)\ (i=1,\ldots,n)$ and $f(x)$ are assumed limited in the indicated interval.

The set of n(linearly independent) solutions of the homogeneous equation (1.1) $\{Y_k(x)\},\ k=0,1,\ldots,n-1,$ satisfying the condition

$$Y_k^{(i)}(a)=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases} \qquad (1.2)$$
$$(i,\ k=0,1,\ldots,n-1),$$

is called the normal fundamental simultaneous equation (1.1) with the initial section x = a.

If it is known that $Y_*(x)$ — is a particular solution of equation (1.1), satisfying zero initial conditions

$$Y_*^{(i)}(a) = 0, \quad i = 0, 1, \ldots, n-1, \tag{1.3}$$

then the solution of the equation is presented as:

$$y(x) = \sum_{k=0}^{n-1} y^{(k)}(a)\, Y_k(x) + Y_*(x), \tag{1.4}$$

where $y^{(k)}(a)$ are values of function y(x) and its first n-1 derivatives in section x = a.

We shall now prove the converse assertion. If the solution of equation (1.1) under arbitrary initial conditions and of the arbitrary function $f(x)$ can be presented in the form

$$y(x) = \sum_{k=0}^{n-1} y^{(k)}(a)\, Z_k(x) + Z_*(x), \tag{1.5}$$

where $Z_*(x) = 0$ at $f(x) = 0$, then function $Z_k(x)$ are the normal **fundamental** functions (homogeneous) of equation (1.1)

$$Z_k(x) = Y_k(x), \tag{1.6}$$

and function $Z_*(x)$ is a particular solution of equation (1.1), satisfying zero initial conditions

$$Z_*(x) = Y_*(x). \tag{1.7}$$

For proof, let us assume at first $f(x) = 0$ [illegible] conditions in such a form:

$$y^{(i)}(a) = \begin{cases} y^{(k)}(a) & i = k, \\ 0 & i \neq k \end{cases} \tag{1.8}$$

Then, from relationship (1.5)

$$y(x) = y^{(k)}(a)\, Z_k(x).$$

By differentiating, we find

$$y^{(i)}(x) = y^{(k)}(a)\, Z_k^{(i)}(x).$$

In view of the dependence of (1.8) at x = a we now obtain

$$Z_k^{(i)}(a)=\begin{cases}1 & i=k,\\ 0 & i\neq k,\end{cases}$$

which proves equality (1.6).

Suppose now the initial conditions are zero:

$$y^{(i)}(a)=0 \qquad (i=0,\ldots,n-1). \tag{1.9}$$

Then from relationship (1.5) it follows

$$y(x)=Z_*(x),$$

but in view of dependence (1.9) equality (1.7) proves to be valid.

We note still a subsequent result. If under the above-indicated conditions

$$y^{(n)}(x)=\sum_{k=0}^{n-1} y^{(k)}(a)\,\Phi_k(x)+\Phi_*(x), \tag{1.10}$$

then

$$\begin{aligned}\Phi_k(x)&=Y_k^{(n)}(x),\\ \Phi_*(x)&=Y_*^{(n)}(x).\end{aligned} \tag{1.11}$$

2. Determination of Normal Fundamental Functions and of a Particular Solution by the Method of Successive Approximations

In designating

$$y^{(n)}(x)=\varphi(x) \tag{2.1}$$

and in considering relationship, being obtained after repeated integration of the equality (2.1),

$$y^{(\nu)}(x)=\sum_{l=0}^{n-\nu-1} y^{(l+\nu)}(a)\,\frac{(x-a)^l}{l!}+$$
$$+\underbrace{\int_a^x\int_a^{x_1}\cdots\int_a^{x_{n-\nu-1}}}_{n-\nu\text{ times}}\varphi(x_{n-\nu})\,dx_{n-\nu}\ldots dx_1$$
$$(\nu=0,1,\ldots,n-1),$$

we shall from equation (1.1)

$$\varphi=N\varphi+\sum_{k=0}^{n-1} y^{(k)}(a)f_k(x)+f(x). \tag{2.2}$$

where

$$N\varphi = -\sum_{l=1}^{n} p_l(x) \underbrace{\int_a^x \ldots \int_a^{x_{l-1}}}_{l \text{ times}} \varphi(x_l)\, dx_l \ldots dx_1, \tag{2.3}$$

$$f_k(x) = -\sum_{l=n-k}^{n} p_l(x) \frac{(x-a)^{l-n+k}}{(l-n+k)!}. \tag{2.4}$$

At $x \to a$ equation (2.2) reverts to an identity by virtue of equality (1.1).

Equation (2.2) represents the normal integral equation*.

Gursa** established it in another form, the solution of equation (2.2) can be presented as:

$$\varphi(x) = \sum_{k=0}^{n-1} y^{(k)}(a)\, \Phi_k(x) + \Phi_*(x). \tag{2.5}$$

In this equality

$$\Phi_k(x) = f_k + Nf_k + N^2 f_k + \ldots = \sum_{s=0}^{\infty} N^s f_k, \tag{2.6}$$

where $N^s f_k$ — signifies $\underline{s}$ times the repeated application of operator N, where $(N^0 f_k = f_k)$.

Correspondingly

$$\Phi_*(x) = f + Nf + N^2 f + \ldots = \sum_{s=0}^{\infty} N^s f. \tag{2.7}$$

The series (2.6) and (2.7) converge uniformly and absolutely.

By virtue of equalities (1.11), formulas (2.6) and (2.7) solve the posed problem.

For a determination of function $Y_k(x)$ and its derivatives one should use equality

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*Theory of normal integral equations is considered in Chapter 3.

**E. Gursa, Course of Mathematical Analysis, Vol. 3, State Theoret. Technical Publ. House. Moscow-Leningrad, 1934

$$Y_k^{(i)}(x) =$$

$$= \begin{cases} \underbrace{\int_a^x \cdots \int_a^{x_{n-i-1}}}_{n-i \text{ times}} \Phi_k(x_{n-i})\, dx_{n-i} \ldots dx_1 & (i>k) \\ 1 + \underbrace{\int_a^x \cdots \int_a^{x_{n-k-1}}}_{n-k \text{ times}} \Phi_k(x_{n-k})\, dx_{n-k} \ldots dx_1 & (i=k) \\ \frac{(x-a)^{k-i}}{(k-i)!} + \underbrace{\int_a^x \cdots \int_a^{x_{n-i-1}}}_{n-i \text{ times}} \Phi_k(x_{n-i})\, dx_{n-i} \ldots dx_1 & (i<k) \\ & (i=0,1,\ldots,n-1). \end{cases} \tag{2.8}$$

Equalities (2.8) make it possible to determine all the elements of a normal fundamental matrix of equation (1.1).

Now we present an example. The equation of stability of rod of constant section on two end knuckle bearings has the form

$$y^{(2)}(x) + a^2 y(x) = 0. \tag{2.9}$$

The initial section a = 0;

$$y^{(2)}(x) = \varphi(x).$$

From equalities (2.3) and (2.4) we obtain

$$N\varphi = -\sum_{i=1}^{2} p_i(x) \underbrace{\int_a^x \int_a^{x_{i-1}}}_{i \text{ times}} \varphi(x_i) dx_i \ldots dx_1 =$$

$$= -a^2 \int_0^x \int_0^{x_1} \varphi(x_2)\, dx_2\, dx_1,$$

$$f_0(x) = -\sum_{i=2}^{2} p_i(x) \frac{(x-a)^{i-2}}{(i-2)!} = -a^2,$$

$$f_1(x) = -\sum_{i=1}^{2} p_i(x) \frac{(x-a)^{i-1}}{(i-1)!} = -a^2 x.$$

Equation (2.2) will be such

$$\varphi(x) = -a^2 \int_0^x \int_0^{x_1} \varphi(x_2)\, dx_2\, dx_1 - y(0) a^2 - y^{(1)}(0) a^2 x.$$

According to equality (2.6)

$$\Phi_0(x) = Y_0^{(2)}(x) = -a^2 + \frac{a^4x^2}{2!} - \frac{a^6x^4}{4!} + \dots =$$
$$= -a^2\left(1 - \frac{a^2x^2}{2!} + \frac{a^4x^4}{4!} \dots\right),$$
$$\Phi_1(x) = Y_1^{(2)}(x) = -a^2x + \frac{a^4x^3}{3!} - \frac{a^6x^5}{5!} + \dots = \tag{2.10}$$
$$= -a\left(ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} \dots\right).$$

For equation (2.9) the accurate solution is

$$Y_0(x) = \cos ax, \quad Y_1(x) = \frac{1}{a}\sin ax$$

and furthermore,

$$Y_0^{(2)}(x) = -a^2\cos ax, \quad Y_1^{(2)}(x) = -a\sin ax. \tag{2.11}$$

By comparing formulas (2.11) and (2.10), we readily note that in given case each term of the series (2.6) represents a corresponding term of the expansion of accurate functions into a power series.

We shall make several remarks of a practical nature.

The magnitudes of $N^s f_k$ should be determined by using approximate methods of calculating the integrals of which the simplest is trapezoidal rule.

In calculating the iterated integrals with a variable upper limit, an essential decrease in computing work is obtained by subdividing the interval into sectors of equal length and use of the "Ring rule" (Table 2).

Into the column are entered three numbers, being encompassed by the arrow; for obtaining the true magnitude of integral the values in the column must be multiplied by $\left(\frac{1}{2}\Delta\right)^n$, where Δ -is the length of sector, n-is the number of integration operations.

For determining $N^s f_k (s = 1, 2, 3, \dots)$ the integral operation N is completed on function $N^{s-1} f_k$, the values of which already are in corresponding column of the calculating table $(N^0 f_k = f_k)$.

Table 2. Diagram of integration on basis of the Ring rule

x	$f(x)$	$\int_a^x f(x_1)dx_1$	$\int_a^x\int_a^{x_1} f(x_2)dx_2dx_1$	$\int_a^x\int_a^{x_1}\int_a^{x_2} f(x_3)dx_3dx_2dx_1$
Множ.(a)		$\frac{1}{2}\Delta$	$\left(\frac{1}{2}\Delta\right)^2$	$\left(\frac{1}{2}\Delta\right)^3$
$x_0=a$	f_0	0	0	0
x_1	f_1	□	□	□
x_2	f_2	□	□	□
x_3	f_3	□	□	□

KEY: (a) factor.

3. Discontinuous Solutions

Let us assume that in the solution of equation (1.1), there are given, in addition to the initial conditions, discontinuities of first order of function y(x) and its n--1 first derivatives in the sections $x=a_j$ $(j=1,\ldots,m)$. Part of given discontinuities can have zero values.

We shall designate

$$y^{(\nu)}(a_j+0)-y^{(\nu)}(a_j-0)=\Delta_j^{(\nu)} \quad \begin{pmatrix}\nu=0,1,\ldots,n-1,\\ j=1,2,\ldots,m\end{pmatrix}. \tag{3.1}$$

The initial values also may be considered as the given discontinuities after assuming

$$\begin{aligned} y(a)&=\Delta_0^{(0)},\\ y^{(1)}(a)&=\Delta_0^{(1)},\\ &\cdots\cdots\\ y^{(n-1)}(a)&=\Delta_0^{(n-1)}.\end{aligned} \tag{3.2}$$

We shall introduce a single discontinuous function S(x,c), determinate by the equality

$$S(x, c)=\begin{cases}0 & x<c, \\ 1 & x>c.\end{cases} \tag{3.3}$$

If $f(x)$ is the arbitrary integrand, then

$$\int_0^x S(x_1,c)f(x_1)\,dx_1 = S(x,c)\int_c^x f(x_1)\,dx_1 \tag{3.4}$$

and furthermore

$$\underbrace{\int_0^x\int_0^{x_1}\cdots\int_0^{x_{\nu-1}}}_{\nu\text{ times}} S(x_\nu,c)f(x_\nu)dx_\nu\ldots dx_1 =$$

$$= S(x,c)\underbrace{\int_c^x\int_c^{x_1}\cdots\int_c^{x_{\nu-1}}}_{\nu\text{ times}} f(x_\nu)\,dx_\nu\ldots dx_1. \tag{3.5}$$

In particular, at $f(x)=1$

$$\int_0^x\int_0^{x_1}\cdots\int_0^{x_{\nu-1}} S(x_\nu,c)\,dx_\nu\ldots dx_1 = S(x,c)\,\frac{(x-c)^\nu}{\nu!}. \tag{3.6}$$

If $\varphi(x)-$ is an arbitrary differentiable function, then

$$\frac{d}{dx}[S(x,c)\varphi(x)] = S(x,c)\,\frac{d\varphi}{dx}(x). \tag{3.7}$$

For a discontinuous function we shall have

$$y(x)=\sum_{j=0}^{m}\Delta_j^{(0)}S(x,a_j)+\int_0^x y^{(1)}(x_1)\,dx_1 \tag{3.8}$$

and further

$$y^{(1)}(x)=\sum_{j=0}^{m}\Delta_j^{(1)}S(x,a_j)+\int_0^x y^{(2)}(x_1)\,dx_1. \tag{3.9}$$

By introducing (3.9) into equality (3.8) and by using dependence of (3.4), we obtain

$$\begin{aligned} y(x) = &\sum_{j=0}^{m}\Delta_j^{(0)}S(x,a_j)+\sum_{j=0}^{m}\Delta_j^{(1)}S(x,a_j)(x-a_j)+ \\ &+\int_0^x\int_0^{x_1} y^{(2)}(x_2)\,dx_2\,dx_1. \end{aligned} \tag{3.10}$$

Successively by applying this method we shall find

$$\begin{aligned} y(x) = &\sum_{k=0}^{n-1}\sum_{j=0}^{m}\Delta_j^{(k)}S(x,a_j)\,\frac{(x-a_j)^k}{k!} + \\ &+\underbrace{\int_0^x\int_0^{x_1}\cdots\int_0^{x_{n-1}}}_{n\text{ times}} y^{(n)}(x_n)\,dx_n\ldots dx_1. \end{aligned} \tag{3.11}$$

In differentiating equality (3.11), we establish

$$y^{(\nu)}(x)=\sum_{k=\nu}^{n-1}\sum_{j=0}^{m}\Delta_j^{(k)}S(x,a_j)\frac{(x-a_j)^{k-\nu}}{(k-\nu)!}+$$
$$+\underbrace{\int_a^x\int_a^{x_1}\dots\int_a^{x_{n-1-\nu}}}_{n-\nu\text{ times}}y^{(n)}(x_{n-\nu})\,dx_{n-\nu}\dots dx_1. \tag{3.12}$$
$$(\nu=0,1,\dots,n-1).$$

In designating $$y^{(n)}(x)=\varphi(x)$$

and by introducing equality (3.12) into equation (1.1), we shall obtain

$$\varphi=N\varphi+\sum_{j=0}^{m}\sum_{k=0}^{n-1}\Delta_j^{(k)}f_{kj}(x)+f(x), \tag{3.13}$$

where $N\varphi$ is given by equality (2.3), and

$$f_{kj}(x)=-\sum_{l=n-k}^{n}S(x,a_j)p_l(x)\frac{(x-a_j)^{l-n+k}}{(l-n+k)!}. \tag{3.14}$$

Equation (3.13) is a normal integral equation, equivalent to the differential equation (1.1) together with initial conditions and given discontinuities of the function itself and its n--1 first derivatives in the sections $x=a_j$ $(j=1,\dots,m)$.

If the indicated discontinuities are absent, then equation (3.13) and (2.2) agree by virtue of equalities (3.2).

The solution of equation (3.13) will be such:

$$\varphi(x)=\sum_{j=0}^{m}\sum_{k=0}^{n-1}\Delta_j^{(k)}\Phi_{kj}(x)+\Phi_*(x), \tag{3.15}$$

where the function $\Phi_{kj}(x)$ are determined by the equality

$$\Phi_{kj}(x)=f_{kj}+Nf_{kj}+N^2f_{kj}+\dots=\sum_{s=0}^{\infty}N^sf_{kj}, \tag{3.16}$$

and the function $\Phi_*(x)$ is given by the previous formula (2.7).

It is possible to establish that

$$\Phi_{kj}(x)=Y_{kj}^{(n)}(x)S(x,a_j) \qquad \begin{pmatrix}k=0,\dots,n-1\\ j=0,\dots,m\end{pmatrix}, \tag{3.17}$$

where $Y_{kj}(x)$ — is the k-th normal fundamental function of the homogeneous equation (1.1) with the initial section $x=a_j$.

For an equation with constant coefficients

$$Y_{kj}(x) = Y_k(x - a_j). \qquad (3.18)$$

Normal fundamental functions $Y_k(x)$ with an initial section x = a can be designated now $Y_{k0}(x)$ $(k=0, \ldots, n-1)$.

All other functions $Y_{kj}(x)$ can be represented, as linear combinations of the functions $Y_k(x)$:

$$Y_{kj}(x) = \sum_{k=0}^{n-1} q_{kj} Y_k(x), \qquad (3.19)$$

where the constant coefficients q_{kj} are determined from n equations n

$$Y_{kj}^{(i)}(a_j) = \begin{cases} 1 & i=k, \\ 0 & i \neq k. \end{cases}$$

The expediency of the method of determining $Y_{kj}(x)$ at $j \geqslant 1$ [by means of the series (3.16) or the equality (3.19)] will be determined by peculiarities of the problem

Solution of equation (1.1) with given discontinuities of function y(x) and its n-1 first derivatives in m sections $x=a_j$ has form

$$y(x) = \sum_{j=0}^{m} \sum_{k=0}^{n-1} S(x, a_j)\, \Delta_j^{(k)} Y_{kj}(x) + Y_*(x). \qquad (3.20)$$

4. Application of the Linear Approximation Method*.

We discuss now another method of determining normal fundamental functions, based also on the solution of equation (2.2).

For determining $\Phi_k(x) = Y_k^{(n)}(x)$ there is solved the equation

$$\varphi = N\varphi + f_k, \qquad (4.1)$$

*Other methods of linear approximation will be considered in Chapter 3.

where the solution of equation $\varphi(x)$ agrees with the function $\Phi_k(x)$. (In determining a particular solution, satisfying zero initial conditions, we proceed from the equation $\varphi = N\varphi + f$).

We shall divide interval of variation of x into a number of small sectors and will designate the boundary sections $a = x_0, x_1, \ldots, x_i, \ldots, x_r = b$. Within limits each sector we shall assume the function $\varphi(x)$ as linear.

For first section $(x_0 < x < x_1)$

$$\varphi(x) = \varphi_0 + k_0(x - x_0),$$

where

$$k_0 = \frac{\varphi_1 - \varphi_0}{x_1 - x_0}.$$

In introducing values of $\varphi(x)$ into equality (4.1), we shall obtain at $x = x_1$

$$\varphi_1 = -\varphi_0 \sum_{i=1}^{n} p_{i1} \frac{(x_1 - x_0)^i}{i!} - k_0 \sum_{i=1}^{n} p_{i1} \frac{(x_1 - x_0)^{i+1}}{(i+1)!} + f_{k1},$$

or, by introducing the value k_0,

$$\varphi_1 = \frac{1}{1 + \sum_{i=1}^{n} p_{i1} \frac{(x_1 - x_0)^i}{(i+1)!}} \left\{ -\varphi_0 \sum_{i=1}^{n} \frac{p_{i1}}{(i+1)!} i (x_1 - x_0)^i + f_{k1} \right\}. \quad (4.2)$$

Here, and henceforth the following abbreviated designations are used:

$$\varphi(x_j) = \varphi_j, \quad p_i(x_j) = p_{ij},$$
$$f_k(x_j) = f_{kj}.$$

The equation (polygonal) of the function $\varphi(x)$, valid within the limits $x_0 < x < x_v$, can be expressed in the following manner:

$$\varphi(x) = \varphi_0 + \sum_{j=0}^{v-1} [S(x, x_j) k_j (x - x_j) - - S(x, x_{j+1}) k_j (x - x_{j+1})], \quad (4.3)$$

where

$$k_j = \frac{\varphi_{j+1} - \varphi_j}{x_{j+1} - x_j},$$

and the single discontinuous functions are determined by the equality (3.3).

In introducing values of $\varphi(x)$ from equality (4.3) in equation (4.1), we shall

obtain at $x = x_\nu$

$$\varphi_\nu = \frac{1}{1-a_{\nu\nu}}\left\{\sum_{j=0}^{\nu-1}\varphi_j a_{\nu j} + f_{1\nu}\right\} \qquad (\nu = 1,2,\ldots,r), \tag{4.4}$$

where

$$a_{\nu j} = -\sum_{i=1}^{n} p_{i\nu}\beta_i(\nu, j). \tag{4.5}$$

In the latter equality

$$\beta_i(\nu, j) = \frac{1}{(i+1)!}\left[\frac{(x_\nu - x_{j-1})^{i+1} - (x_\nu - x_j)^{i+1}}{x_j - x_{j-1}} - \frac{(x_\nu - x_j)^{i+1} - (x_\nu - x_{j+1})^{i+1}}{x_{j+1} - x_j}\right] \qquad (1 \leqslant j \leqslant \nu - 1, \quad 1 \leqslant i \leqslant n). \tag{4.6}$$

For $j = 0$ and $j = \nu$ we shall have other formulas:

$$\beta_i(\nu, 0) = \frac{1}{(i+1)!}\left[(i+1)(x_\nu - x_0)^i - \frac{(x_\nu - x_0)^{i+1} - (x_\nu - x_1)^{i+1}}{x_1 - x_0}\right], \tag{4.7}$$

$$\beta_i(\nu, \nu) = \frac{1}{(i+1)!}(x_\nu - x_{\nu-1})^i. \tag{4.8}$$

Let us note that coefficients $a_{\nu j}$ remain identical during determination of all fundamental functions and of the particular solution.

Of prime practical interest is the subdivision of interval into equal sectors.

Here

$$x_\nu - x_j = \Delta(\nu - j), \tag{4.9}$$

where Δ is the length of a section.

Equalities (4.6), (4.7), and (4.8) now will acquire the form

$$\beta_i(\nu, j) = \frac{\Delta^i}{(i+1)!}\left[(\nu - j + 1)^{i+1} - 2(\nu - j)^{i+1} + (\nu - j - 1)^{i+1}\right] \qquad (1 \leqslant j \leqslant \nu - 1, \quad 1 \leqslant i \leqslant n), \tag{4.10}$$

$$\beta_i(\nu, 0) = \frac{\Delta^i}{(i+1)!}\left[\nu^i(1+i) - \nu^{i+1} + (\nu - 1)^{i+1}\right], \tag{4.11}$$

$$\beta_i(\nu, \nu) = \frac{\Delta^i}{(i+1)!}. \tag{4.12}$$

The latter formulas are conveniently presented as

$$\beta_i(\nu, j) = \frac{\Delta^i}{(i+1)!}\gamma_i(\nu, j).$$

where the coefficients $\gamma_i(\nu, j)$ remain one and the same for any (linear) differential equations.

We now present, as an example, these coefficients for an equation of fourth order and for subdivision of the interval into ten sectors (Table 3). The matrix of coefficients $\gamma_i(\nu, j)$ is triangular. All elements of the main diagonal are separated by a heavy line, equal to unity.

Elements, standing in secondary diagonals, with the exception of those belonging to the first column, are identical; therefore, calculation will subject only the elements of first two columns. The tables remain valid also in the subdivision into a smaller number of sectors.

Below there is presented a diagram of the calculation (Table 4). The first part of the table contains the values $a_{\nu j}$, in which the coefficients $a_{\nu\nu}$ are differentiated by a heavy line. The values $f_{k\nu}$ are known. At $\nu=0$ $\varphi_0=f_{k0}$. Furthermore the magnitude φ_0 is multiplied by elements of column 0 in the table $a_{\nu j}$ and is entered into the column 0 of table $a_{\nu j}\varphi_j$.

For obtaining φ_ν there are summarized all the terms, standing in the line ν, and they are divided by the magnitude $1-a_{\nu\nu}$. After obtaining φ_ν column ν is filled in the table $a_{\nu j}\varphi_j$ et cetera.

The values $Y_k^{(i)}(x)$ $(i=0,1,\ldots,n-1)$ are determined from relationships (2.8). Since here the integral operations are completed, then the accuracy of calculation increases.

In the considered methods of approximate integration of differential equations there exists an effective method of verification.

The approximate value $\varphi(x)$ is introduced into the integral of equation (2.2) and there is determined the difference between the left and right sides which constitutes the error of the solution $\varepsilon(x)$.

By integrating $\varepsilon(x)$ in accordance with equalities (2.8), we find error in function $Y_k(x)$ and its derivatives. From physical considerations there is

Table 3. Table of coefficients $\gamma_i(\nu, j)$ (for brevity the j columns have not been extended).

	$i=1$				$i=2$				$i=3$				$i=4$					
$\nu \backslash j$	0	1	2		0	1	2		0	1	2		0	1	2			
0																		
1	1	1			2	1			3	1			4	1				
2	1	2	1		5	6	1		17	14	1		49	30	1			
3	1	2	2		8	12	6		43	50	14		194	80	30			
4	1	2	2		11	18	12		81	110	50		499	570	180			
5	1	2	2		14	24	18		131	194	110		1024	1320	570			
6	1	2	2		17	30	24		193	302	194		1829	2550	1320			
7	1	2	2		20	36	30		267	434	302		2974	4380	2550			
8	1	2	2		23	42	36		353	590	434		4519	6930	4350			
9	1	2	2		26	48	42		451	770	590		6524	10320	6930			
10	1	2	2		29	54	48		561	974	770		9049	11670	10320			

Table 4. Diagram of calculation by the linear approximation method.

x		$a_{\nu j}$						$a_{\nu j}\varphi_j$						$f_{k\nu}$	Σ	φ_ν
	$\nu \backslash j$	0	1	2	3		r	0	1	2	3		$r-1$			
x_0	0													f_{k0}		$\varphi_0=f_{k0}$
x_1	1	a_{10}	a_{11}					$\varphi_0 a_{10}$						f_{k1}		φ_1
x_2	2	a_{20}	a_{21}	a_{22}				$\varphi_0 a_{20}$						f_{k2}		
x_3	3	a_{30}	a_{31}	a_{32}	a_{33}			$\varphi_0 a_{30}$						f_{k3}		
x_r	r	a_{r0}	a_{r1}	a_{r2}	a_{r3}		a_{rr}	$\varphi_0 a_{r0}$						f_{kr}		

established the admissibility of any one error.

We note that a majority of engineering problems errors of an order of 2 to 5% are entirely admissible, since they correspond to the accuracy of the given initial magnitudes.

5. Method of Moving Origin

The effectiveness of previously presented methods decreases with an increase in the length of interval of x variation, in the elongation of which the solution is sought.

This circumstance is peculiar to almost all methods of approximate integration of differential equations.

In using the method of successive approximations the convergence for large x values deteriorates.

Thus, for example, in equalities (2.10) there is obtained an expansion by degrees of x, which converges at any x values, but a small number of approximations gives good accuracy only at $ax < \sqrt{2}$.

For the method of linear approximation the accuracy increases with a decrease in the length of sector of interpolation, but an increase in the number of sections results in a large increase of computing work, the indicated reasons make expedient the application of a special method of calculation, to the discussion of which we now turn.

We shall consider at first the method of successive approximations.

Suppose, for example, there is determined the k-th normal fundamental function, i.e., equation (4.1) is solved.

$$\varphi = N\varphi + f_k.$$

Furthermore from the calculation it is ascertained that with three to four approximations the values φ for $x < a_1$ agree with the required accuracy. Then, by means of formulas (2.8) all $y^{(k)}(a_1)$ $(k = 0, \ldots, n-1)$ are found. Now it is

possible to transfer the initial section into $x = a_1$ and to make the calculation according to the equation

$$\varphi = N_1\varphi + \sum_{k=0}^{n-1} y^{(k)}(a_1) f_k(x), \tag{5.1}$$

where

$$N_1\varphi = -\sum_{l=1}^{n-1} p_l(x) \underbrace{\int_{a_1}^{x} \cdots \int_{a_1}^{x_{l-1}}}_{l\ \text{times}} \varphi(x_l)\,dx_l \ldots dx_1,$$

$$f_k(x) = -\sum_{l=n-k}^{n} p_l(x) \frac{(x-a_1)^{l-n+k}}{(l-n+k)!};$$

thus equality (2.2) is valid at an arbitrary value a.

Briefly written, equation (5.1) has the form

$$\varphi = N_1\varphi + f_{1k},$$

where

$$f_{1k} = \sum_{k=0}^{n-1} y^{(k)}(a_1) f_k(x)$$

is a known function. The subsequent stage of the calculation repeats the preceding.

For determining the particular solution $\Phi_0(x)$ there is solved the equation

$$\varphi = N\varphi + f.$$

In second section

$$\varphi = N_1\varphi + \sum_{k=0}^{n-1} y^{(k)}(a_1) f_k(x) + f(x),$$

where $N_1\varphi$ and $f_k(x)$ are the same as in equality (5.1).

We now establish an evaluation making it possible to determine a_1 prior to completion of calculation. The matter reduces to an evaluation of terms in the series (2.6) or (2.7).

We shall give a very "rigid" evaluation, which will assure the condition of rapid convergence in the process of successive approximations.

Suppose

$$A = \max_{a<x<b} \begin{cases} |f_k(x)|, \\ |f(x)|, \end{cases} \tag{5.2}$$

$$P_l = \max_{a<x<b} |p_l(x)|.$$

Then

$$|Nf_k| \leqslant \sum_{l=1}^{n} P_l \int_a^x \dots \int_a^{x_{l-1}} |f_k(x_l)|\, dx_l \dots dx_1 \leqslant A \sum_{l=1}^{n} P_l \frac{l_1^l}{l!}. \tag{5.3}$$

where $\qquad l_1 = a_1 - a,$

$$|N^2 f_k| \leqslant \sum_{l=1}^{n} P_l \int_a^x \dots \int_a^{x_{l-1}} |Nf_k|\, dx_l \dots dx_1 \leqslant$$

$$\leqslant A\left(\sum_{l=1}^{n} P_l \frac{l_1^l}{l!}\right)^2 \tag{5.4}$$

and in general

$$|N^s f_k| \leqslant A\left(\sum_{l=1}^{n} P_l \frac{l_1^l}{l!}\right)^s.$$

Under the condition

$$\sum_{l=1}^{n} P_l \frac{l_1^l}{l!} < 1 \tag{5.5}$$

series (2.6) and (2.7) will be rapidly convergent which can be established by comparing an evaluation made more strictly.

From the latter it follows, as already was indicated, that the series (2.6) and (2.7) are absolutely and uniformly convergent for the finite values A and P_l, in which one is readily convinced after presenting N in the canonical form of a Volterra operator.

We now present some examples, relating to equation (2.9). For determining $\Phi_0(x)$ we solve the equation

$$\varphi(x) = -a^2 \int_0^x \int_0^{x_1} \varphi(x_2)\, dx_2\, dx_1 - a^2.$$

we shall have

$$A = a^2, \quad P_2 = a^2.$$

Evaluations of (5.3) and (5.4) give

$$|Nf_k| \leqslant a^4 \frac{l_1^2}{2}, \tag{5.6}$$

$$|N^2 f_k| \leqslant a^6 \frac{l_1^4}{4}.$$

condition of convergence (5.5)

$$\frac{a^2 l_1^2}{2} < 1, \quad a l_1 < \sqrt{2}. \tag{5.7}$$

The more accurate evaluation

$$|N f_k| \leqslant a^2 \frac{x^2}{2!} \quad |N^2 f_k| \leqslant a^4 \frac{x^4}{4!} \tag{5.8}$$

established convergence of process of successive approximations in an arbitrary, but finite interval of variation of x.

At $x = l_1$, real value $|N^2 f_k|$ is 6 times less than the evaluation (5.6).

Condition (5.7) assures a rapid convergence of the process of successive approximations in the interval $a < x < l_1$.

However, in practice, an accurate evaluation of the magnitude a_1 (limits of rapid convergence), is not required since it is ascertained in the process of calculation with an arbitrary value a_1; it may be found only that some of calculations at large x values, will appear to be superfluous.

This is virtually established after the first two approximations.

We note that selection of magnitude a_1 and "limiting" number of utilized approximations is determined by peculiarities of the operator $N\varphi$. Obviously, the more simple the structure of the operator is the greater number of approximations can be applied and the rarer can the transfer of origin be used.

Method of moving origin without any changes is extended to the method of linear approximation, where the use of more than ten sectors becomes unwieldy.

6. Quasinormal Fundamental Functions

The differential equation of the type

$$\frac{d^{\nu_j}}{dx^{\nu_j}}\left\{p_j(x) \; . \; . \; . \; \frac{d^{\nu_1}}{dx^{\nu_1}}\left[p_1(x)\frac{d^{\nu_0}}{dx^{\nu_0}} y(x)\right]\right\} - q(x)\,y(x) = f(x) \tag{6.1}$$

we shall call an inhomogeneous binomial equation; correspondingly at f(x) = 0, it is homogeneous. To equations of this type belong a large number of equations, encountered in engineering problems.

We shall assume that

$$p_i(x) \neq 0$$

at $1 < i < j,\ a < x < b.$

The sum $\nu_0 + \nu_1 + \ldots + \nu_j = n$ determines order of equation (6.1).

We call the magnitude

$$\frac{d^{\nu_0}}{dx^{\nu_0}}\left\{p_s(x)\ \ldots\ \frac{d^{\nu_1}}{dx^{\nu_1}}\left[p_1(x)\frac{d^{\nu_0}}{dx^{\nu_0}}y(x)\right]\right\} = y^{[m]}(x) \qquad (6.2)$$

a quasiderivative of the function of order m, where

$$m = \nu_0 + \nu_1 + \ldots + \nu_s.$$

The designation of a quasiderivative is supplied by the superscript in brackets.

In writing out equation (6.1) in the form

$$\frac{d^{\nu_j}}{dx^{\nu_j}}\left\{p_j(x)\ \ldots\ \frac{d^{\nu_1}}{dx^{\nu_1}}\left[p_1(x)\frac{d^{\nu_0}}{dx^{\nu_0}}y(x)\right]\right\} = q(x)\,y(x) + f(x)$$

and integrating both sides of equality n times $(n = \nu_0 + \ldots + \nu_j)$ with the limits from a to x, we obtain

$$y = Ny + \sum_{k=0}^{n-1} y^{[k]}(a)F_k(x) + F(x), \qquad (6.3)$$

where

$$Ny = \underbrace{\int_a^x\int_a^{x_1}\ldots\int_a^{x_{\nu_0-1}}}_{\nu_0\ \text{times}}\frac{1}{p_1(x_{\nu_0-1})}\int_a^{x_{\nu_0}}\ldots\int_a^{x_{\nu_0+\nu_1-1}}\frac{1}{p_2(x_{\nu_0+\nu_1-1})}$$

$$\underbrace{\int_a^{x_{n-\nu_j-1}}\ldots\int_a^{x_{n-1}}}_{\nu_j\ \text{times}} q(x_n)\,y(x_n)\,dx_n\ldots dx_1, \qquad (6.4)$$

$$F_k(x) = \frac{(x-a)^k}{k!}, \qquad (6.5)$$

$$F(x) = \int_a^x\ldots\int_a^{x_{\nu_0-1}}\frac{1}{p_1(x_{\nu_0-1})}\ldots \qquad (6.6)$$

$$\ldots\int_a^{x_{n-\nu_j-1}}\ldots\int_a^{x_{n-1}} f(x_n)\,dx_n\ldots dx_1.$$

Quasinormal fundamental functions are determined by the series

$$Y_k(x) = F_k + NF_k + N^2F_k + \ldots \qquad (k = 0, 1, \ldots, n-1). \qquad (6.7)$$

These functions satisfy the equality

$$Y_k^{[i]}(a) = \begin{cases} 1 & i=k, \\ 0 & i \neq k. \end{cases} \tag{6.8}$$

The particular solution of equation (6.1), satisfying zero initial conditions, is expressed by the series

$$Y_*(x) = F + NF + N^2F + \dots \tag{6.9}$$

As an example we shall consider equation of flexure of a beam of variable section on elastic foundation:

$$\frac{d^2}{dx^2}\left[EJ(x)\frac{d^2y}{dx^2}\right] + k(x)y(x) = f(x). \tag{6.10}$$

Here $y(x)$ is the sag of axis of beam;
$EJ(x)$ is the strength of beam to flexure bend;
$k(x)$ is the coefficient of elasticity of foundation;
$f(x)$ is the distributed load per unit of length of beam.

From equation (6.3) we shall have

$$y(x) = -\int_0^x\int_0^{x_1}\frac{1}{EJ(x_2)}\int_0^{x_2}\int_0^{x_3} k(x_4)y(x_4)\,dx_4\,dx_3\,dx_2\,dx_1 + \\ + \sum_{k=0}^{3} y^{[k]}(0)F_k(x) + F(x), \tag{6.11}$$

where

$$F_k(x) = \frac{(x-a)^k}{k!}, \quad F(x) = \int_0^x\int_0^{x_1}\frac{1}{EJ(x_2)}\int_0^{x_2}\int_0^{x_3} f(x_4)\,dx_4 \dots dx_1.$$

Quasiderivatives have following physical meaning:

$$y^{[0]}(x) = y(x); \quad y^{[1]}(x) = y^{(1)}(x);$$
$$y^{[2]}(x) = y^{(2)}(x) = \frac{M(x)}{EJ}; \quad y^{[3]}(x) = \frac{d}{dx}\left(EJ\frac{d^2y}{dx^2}\right) = Q(x).$$

where $M(x)$ and $Q(x)$ are the bending moment and transverse force in section x.

The solution of equation (6.11) is written out as:

$$y(x) = \sum_{k=0}^{3} y^{[k]}(0)Y_k(x) + Y_*(x). \tag{6.12}$$

where the functions $Y_k(x)$ and $Y_*(x)$ are expressed by the converging series (6.7) and (6.9).

Equation (6.11) may also be solved by the method of linear approximation.

As previously, in a number of cases it is expedient to use the method of the mobile origin section.

Integral equation (6.3) may be applicable for the solution of nonlinear equations of the form

$$\frac{d^{\nu_j}}{dx^{\nu_j}}\left\{p_j(x)\frac{d^{\nu_1}}{dx^{\nu_1}}\left[p_1(x)\frac{d^{\nu_0}}{dx^{\nu_0}}y(x)\right]\right\}+\varphi(x,y)=f(x). \tag{6.13}$$

To an equation of this type, belongs the well-known equation of M. V. Ostrogradskiy

$$\frac{d^2y}{dx^2}+\alpha y+\beta y^3=0, \tag{6.14}$$

which is a subject of analysis in works of a number of outstanding mathematicians.

Equation (6.3) remains in force, if only in equality (6.4) we replace $q(x)\dot{y}(x)$ by $\varphi(x, y)$.

We note that the application of method of successive approximations together with method of moving origin gave a solution of equation (6.14), entirely satisfactory for engineering applications.

CHAPTER 3

BOUNDARY AND NORMAL INTEGRAL EQUATIONS

Modified Fredholm and Volterra integral equations, which are called boundary and normal integral equations are considered.

Origin of these terms will be clear from the discussion later on.

It is possible to show that the boundary integral operator reduces to a Fredholm operator, i.e., it is expressed in form

$$Ky=\int_a^b G(x,s)y(s)ds,$$

and normal operator is equivalent to the Volterra operator, but the presentation of considered integral equations in form of classical integral equations frequently is difficult, and the principal--completely unnecessary from the point of view of practical use.

It is necessary also to consider that the origination of Fredholm or Volterra equations frequently dense is found to be a very complicated matter, whereas boundary and normal integral equations naturally ensue from differential equations.

For an illustration of statement above it suffices to turn to the problem about stability of a rod, for which construction of Fredholm equation requires a number of artificial reasonings and computations. It becomes intelligible, why the classical Fredholm and Volterra integral equations which proved to be a very effective apparatus for a general and qualitative investigation, were not widely used in solving engineering problems.

Boundary integral equations were used also earlier for solution of theoretical and applied questions.

Usually these equations ensued as the result of application of method of successive approximations during the solution of differential equations. However, with such an approach there was lost the generality, peculiar to the apparatus of integral equations.

As an example, it is possible to point to the method of successive approximations in problems of stability (Vianello method), which is a combination of particular methods including, graphical-analytic operations.

We shall present another example.

For calculating a beam on an elastic foundation A. N. Krylov used process successive approximations, which can be represented as the solution of a boundary integral equation

$$y = \lambda K y + f$$

by the method of simple iteration. Parameter of equation $\lambda = -1$.

It is easy to establish that the homogeneous equation $y = \lambda K y$ corresponds to the problem on the vibration of a beam with a certain distributed "masses" and has all positive eigenvalue values $\lambda_1, \lambda_2, \ldots$. The process of simple iteration is convergent at $\left|\frac{\lambda}{\lambda_1}\right| < 1$.

A. N. Krylov detected the divergence of the process only by a direct analysis of the obtained series. For the case $|\lambda| < |\lambda_1|$ in the work "On Calculating Beams, Lying on an Elastic Foundation" the process of successive approximations, is proposed: it did not give, however, satisfactory results.

Meanwhile, the use of theory of integral equations makes it possible to construct effective convergent processes, to establish a comprehensive generality between problems on strength, vibration and stability of rods.

Works of the outstanding scientist A. N. Krylov have promoted development of method of boundary integral equations.

In article by P. F. Papkovich[*], which continues the work of A. N. Krylov[**], there is indicated the process of determining the eigenfunctions and eigenvalues now widely used, somewhat earlier a similar method was used by V. P. Vetchinkin[***].

The method of boundary integral equations in the works of E. P. Grossman, D. Yu. Panov, P. M. Reese, and S. A. Tumarkin, is further developed.

It is necessary to note the works of Sh. E. Mikeladze, in which there are widely used the Volterra equation and in individual cases, a transition to normal integral equations is observed.

A consideration of boundary and standard equations as a general mathematical device for the first time was done by the Soviet scientist Yu. V. Repman[****].

Equations, similar to the considered equations, Yu. V. Repman called "equations in indefinite integrals".

In the present chapter there are considered elements of theory of boundary and normal integral equations, there are indicated methods of solving homogeneous and in homogeneous boundary and normal integral equations. Much attention will be given to systems of integral equations, which are presented in the form of matrix integral equations.

The considered methods can be applied to any engineering problems, which reduce to ordinary differential equations or their systems, and also to partial differential equations, which reduce to ordinary after a separation of variables.

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*P. F. Papkovich, Concerning the Question of Applicability of the Process Successive Approximations for the Flexure of Beams on an Elastic Foundation, "Applied Mathematics and Mechanics", Vol. 1, No. 2, 1933.

**A. N. Krylov, On Calculating Beams, Lying on an Elastic Foundation, Academy of Sciences of USSR, Moscow, 1931.

***V. P. Vetchinkin, Theory of Screw Propellers, Moscow, Zhukovskiy, V. VIA Publ. House, 1926.

****Yu. V. Repman, On Determining Critical Forces by Equations of Stability, Transactions of Laboratory of Engineering Mechanics, Engineering Publ. House, Moscow, 1942.

1. Classification of Equations

Equations of the form

$$y = \lambda Ky + \sum_{i=0}^{m} f_i L_i y + f, \tag{1.1}$$

where y is an unknown function x; Ky is a linear integral operator; $L_i y$ is a linear functional; f_i and f are the functions x (in the interval $a<x<b$); λ is a parameter of the equation, we shall call on one-parameter integral equation. Here and henceforth there are considered only real values of the functions and of the independent variable.

In a general case the operator Ky can be presented in the form of a table

$$\begin{aligned}
Ky = {} & q_{011} \int_{a_{011}}^{x} q_{111} y(x_1)\, dx_1 + q_{012} \int_{a_{012}}^{x} q_{112} \int_{a_{112}}^{x_1} q_{212} y(x_2)\, dx_2\, dx_1 + \dots \\
& \dots + p_{011} \int_{a_{011}}^{b_{011}} p_{111} y(x_1)\, dx_1 + p_{012} \int_{a_{012}}^{b_{012}} p_{112} \int_{a_{112}}^{x_1} p_{212} y(x_2)\, dx_2\, dx_1 + \dots \\
& \dots + q_{021} \int_{a_{021}}^{x} q_{121} y(x_1)\, dx_1 + q_{022} \int_{a_{022}}^{x} q_{122} \int_{a_{122}}^{x_1} q_{222} y(x_2)\, dx_2\, dx_1 + \dots \\
& \dots + p_{021} \int_{a_{021}}^{b_{021}} p_{121} y(x_1)\, dx_1 + p_{022} \int_{a_{022}}^{b_{022}} p_{122} \int_{a_{122}}^{x_1} p_{222} y(x_2)\, dx_2\, dx_1 \dots
\end{aligned} \tag{1.2}$$

In this equality q_{rsk} and p_{rsk} are given functions of <u>x</u>, a_{rsk} and b_{rsk} are constant numbers. The subscript "r" is connected with place of function or parameter in the integral expressions; the subscript "s" indicates the line in tabular writing (1.2): the subscript "k" is equal to the number of column. After the line, containing the function <u>q</u>, comes the analogous line (with the same number), containing function <u>p</u>.

In a brief form, first form of presentation of integral operator [equality (1.2)] will be:

$$\begin{aligned}
Ky = \sum_{s=1}^{n_1} \sum_{k=1}^{m_1} \Bigg(& q_{0sk} \int_{a_{0sk}}^{x} q_{1sk} \dots q_{k-1,s,k} \int_{a_{k-1,s,k}}^{x_{k-1}} q_{ksk} y(x_k)\, dx_k \dots dx_1 + \\
& + p_{0sk} \int_{a_{0sk}}^{b_{0sk}} p_{1sk} \dots p_{k-1,s,k} \int_{a_{k-1,s,k}}^{x} p_{ksk} y(x_k)\, dx_k \dots dx_1 \Bigg).
\end{aligned} \tag{1.3}$$

Formulas (1.2) and (1.3) express a boundary integral operator. If all the constant limits of integrations are identical

$$a_{rsk} = b_{0sk} = a,$$

then the integral operator (1.3) we agree to call normal

$$Ny = \sum_{s=1}^{m_1} \sum_{k=1}^{m_1} q_{0sk} \int_a^x q_{1sk} \cdot \cdot \cdot q_{k-1,s,k} \int_a^{x_{k-1}} q_{ksk} y(x_k)\, dx_k \cdot \cdot \; dx_1. \tag{1.4}$$

In accordance with this we shall differentiate the boundary integral equation (equality 1.1) and normal integral equations

$$y = \lambda Ny + \sum_{i=0}^{m} f_i L_i y + f. \tag{1.5}$$

From the preceding it is clear that the normal equations are a particular case of the boundary, similarly as the integral Volterra equations are a particular case of Fredholm equations; however, essential peculiarities of the normal equations make a separate examination of them expedient.

Equations (1.1) and (1.5) contain linear functionals $L_i y$, i.e., parameters, depending on y. As $L_i y$ usually there are used values of the function $y(x)$ or its derivatives at certain points $(x = a_i)$ or values in fixed sections of the integral expressions, entering into Ky.

The selection of functionals for a boundary equation is not obligatory, since they in essence already are contained in boundary integral operator. Therefore, as the basic form of the boundary integral equation it is possible to adapt the following:

$$y = \lambda Ky + f. \tag{1.6}$$

Solution of the equation satisfies all boundary conditions of the problem.

In rarer cases, the boundary integral equations are used in general form (1.1). For normal integral equation, the general form is given by the equality (1.5).

Let us present an example.

The differential equation for the stability of rod of variable section, supported at ends on hinges, has the form (Fig. 8).

$$\frac{d^2y}{dx^2}(x) + \frac{P}{EJ(x)}\, y(x) = 0, \tag{1.7}$$

Fig. 8. Stability of rod.

where y(x) is the sag of axis of rod; EJ(x)--strength of section of rod to flexure.

From equation (1.7) we obtain

$$\frac{dy}{dx}(x) = -P\int_0^x \frac{y(x_1)}{EJ(x_1)}\,dx_1 + \frac{dy}{dx}(0).$$

By repeating the operation of integration, we find

$$y(x) = -P\int_0^x\int_0^{x_1} \frac{y(x_2)}{EJ(x_2)}\,dx_2 dx_1 + x\frac{dy}{dx}(0). \qquad (1.8)$$

The obtained equation is a normal integral equation for the stability of the rod.

If one were to determine $\frac{dy}{dx}(0)$ from a boundary condition $y(l)=0$, then, we arrive at the boundary integral equation

$$y(x) = P\left(\frac{x}{l}\int_0^l\int_0^{x_1} \frac{y(x_2)}{EJ(x_2)}\,dx_2\,dx_1 - \int_0^x\int_0^{x_1} \frac{y(x_2)}{EJ(x_2)}\,dx_2\,dx_1\right). \qquad (1.9)$$

Operator Ky, entering into equation (1.1), is linear, i.e., bounded operator, possessing property of additivity:

$$K(y_1+y_2) = Ky_1 + Ky_2, \qquad (1.10)$$

where y_1 and y_2 are arbitrary integrands.

Boundedness (and, consequently, the continuity) of the operator ensues from the fact that all the functions q_{rsk} and p_{rsk} in equality (1.2) are assumed to be bounded.

The linear operator is also homogeneous:

$$K(\mu y) = \mu K y,$$

where μ is an arbitrary parameter.

Also the functionals, entering into equation (1.1) also possess analogous properties.

A boundary or normal integral equation we call homogeneous, if it admits a

trivial solution

$$y(x) \equiv 0. \tag{1.11}$$

Thus, for example, equation (1.8) and (1.9) are homogeneous. By virtue of the homogeneity of operators and functionals, entering into equation (1.5) and (1.6), the latter will be homogeneous only in the case, if

$$f \equiv 0.$$

In solving homogeneous boundary, equations

$$y = \lambda K y$$

is determined spectrum of eigenvalues $\{\lambda_i\}$ and their corresponding eigenfunctions $\{y_i\}$. A homogeneous normal equation of the form

$$y = \lambda N y$$

does not possess any other solutions, except an identity equal to zero.

In solving inhomogeneous equations (boundary and normal) the parameter λ is given.

In a number of cases it is convenient to use the second form of integral operator:

$$Ky = \sum_{s=1}^{n} \left(Q_s \int_{a_s}^{x} q_s y(x_1)\, dx_1 + P_s \int_{a_s}^{b_s} p_s y(x_1)\, dx_1 \right), \tag{1.12}$$

where Q_s, q_s, P_s and p_s are given functions of x,

a_s and b_s are constant numbers.

With identical constant limits of integration we shall have a second form of a normal integral operator:

$$Ny = \sum_{s=1}^{n} Q_s \int_{a}^{x} q_s y(x_1)\, dx_1 \tag{1.13}$$

As a rule, more simply the integral equation is obtained with the first form of operator; this form is more convenient also in solving the integral equation by method of successive approximations. The second form has the advantage with the use of certain other methods of solution (for example, methods of approximation).

In practical problems there may be encountered systems of integral equations, which expediently are presented in matrix form. Thus, a matrix boundary value integral equation has the form

$$[y]=\lambda[K][y]+[f], \tag{1.14}$$

where the matrices-columns of the unknown and given functions

$$[y]=\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \quad [f]=\begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}, \tag{1.15}$$

and boundary value operator

$$[K][y]=\lambda\begin{bmatrix} K_{11} & \dots & K_{1n} \\ K_{21} & \dots & K_{2n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K_{n1} & \dots & K_{nn} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix}. \tag{1.16}$$

Finally, the matrix integral equations of the following structure are of interest*:

$$\begin{bmatrix} y \\ y^{(1)} \\ \cdot\ \cdot \\ y^{(\nu)} \end{bmatrix}=\lambda\begin{bmatrix} K_{00} & \dots & K_{0\nu} \\ K_{10} & \dots & K_{1\nu} \\ \cdot & \cdots & \cdot \\ K_{\nu 0} & \dots & K_{\nu\nu} \end{bmatrix}\begin{bmatrix} y \\ y^{(1)} \\ \cdot\ \cdot \\ y^{(\nu)} \end{bmatrix}+\begin{bmatrix} f_0 \\ f_1 \\ \cdot \\ f_\nu \end{bmatrix} \tag{1.17}$$

or in short form

$$[y]=\lambda[K^{(\nu)}][y]+[f].$$

Here matrix-column $[y]$ will be formed by function y(x) and its derivatives up to order ν inclusively.

Equations of the form (1.17) are encountered, for example, during calculation of shafts for the critical speed with a calculation of the gyroscopic effect of the distributed masses.

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*Equations of this form may be called integro-differential. However, their distinction from integral equations is immaterial.

2. Formation of Integral Equations from Differential Equations.

Suppose we have a linear differential equation of n-th order with variable coefficients, given in the closed interval $a < x < b$:

$$y^{(n)}(x) + p_1(x) y^{(n-1)}(x) + \ldots + p_n(x) y^{(0)}(x) = f(x) \qquad (2.1)$$

with linear boundary value conditions of general form

$$\sum_{\nu=0}^{n-1} (\alpha_{\nu k} y^{(\nu)}(a) + \beta_{\nu k} y^{(\nu)}(b)) = \gamma_k \qquad (2.2)$$

$$(k = 0, 1, \ldots, n-1).$$

if

$$\alpha_{\nu k} = \begin{cases} 1 & \nu = k, \\ 0 & \nu \neq k, \end{cases} \quad \beta_{\nu k} \equiv 0$$

$$(\nu, k = 0, 1, \ldots, n-1),$$

then conditions (2.2) are Cauchy conditions (at x = a there is given the value of the function and its n-1 first derivatives).

As the fundamental variable in composing the integral equation we shall take

$$y^{(n)}(x) = \varphi(x).$$

In considering equalities

$$y^{(n-1)}(x) = y^{(n-1)}(a) + \int_a^x \varphi(x_1)\, dx_1 \text{ et cetera} \qquad (2.3)$$

we obtain from equation (2.1)

$$\varphi = N\varphi + \sum_{k=0}^{n-1} y^{(k)}(a) f_k(x) + f(x), \qquad (2.4)$$

where

$$N\varphi = -\sum_{i=1}^{n} p_i(x) \int_a^x \ldots \int_a^{x_{i-1}} \varphi(x_i)\, dx_i \ldots dx_1, \qquad (2.5)$$

$$f_k(x) = -\sum_{i=n-k}^{n} p_i(x) \frac{(x-a)^{i-n+k}}{(i-n+k)!}. \qquad (2.6)$$

Equation (2.4) is a normal integral equation.

In another form (viz., in the form a Volterra equation), it was encountered

earlier*. Equivalence of normal integral equation and Volterra equation in a given case is readily established by means of a Dirichlet identity.

$$\underbrace{\int_a^x\int_a^{x_1}\cdots\int_a^{x_{i-1}}}_{i\ \text{times}}\varphi(x_i)\,dx_i\ldots dx_1=\int_a^x\frac{(x-s)^{i-1}}{(i-1)!}\varphi(s)\,ds.$$

Owing to the linearity of operator $N\varphi$ the solution of equation (2.4) in a general case can be presented in following form:

$$\varphi=\sum_{k=0}^{n-1}y^{(k)}(a)\Phi_k(x)+\Phi_*(x), \tag{2.7}$$

where the function $\Phi_k(x)$ is the solution of equation

$$\varphi=N\varphi+f_k, \quad (k=0,1,\ldots,n-1), \tag{2.8}$$

and function $\Phi_*(x)$ satisfies the equation

$$\varphi=N\varphi+f. \tag{2.9}$$

Suppose $\{Y_k(x)\}$ $(k=0,1,\ldots,n-1)$ is the sequence of normal fundamental functions of equations (2.1) and $Y_*(x)$ is the particular solution of this equation at zero initial conditions. It is possible to show the validity of the equalities

$$\begin{aligned}\Phi_k(x)&=Y_k^{(n)}(x),\\ \Phi_*(x)&=Y_*^{(n)}(x).\end{aligned} \tag{2.10}$$

Thus, the solution of the normal integral equation results in a determination of normal fundamental functions of the corresponding linear differential equation.

If the initial Cauchy conditions are given, the function on right side of equation (2.4) is known, then,the solution of the normal integral equation (2.4) determines the solution of differential equation (2.1), satisfying the indicated conditions. If there are given boundary conditions of a general form [condition (2.2)], then by using equality

*E. Gursa, Course of Mathematical Analysis, Vol. III, Moscow-Leningrad. State Theor Tech Publ. House, 1934; Sh. E. Mikeladze, Certain Problems of Structural Mechanics, Moscow, State Engineer.Publ. House, 1948.

$$y^{(\nu)}(b) = \sum_{l=0}^{n+\nu-1} y^{(l+\nu)}(a)\frac{(b-a)^l}{l!} +$$

$$+\int_a^b\int_a^{x_1}\ldots\int_a^{x_{n-\nu-1}}\varphi(x_{n-\nu})\,dx_{n-\nu}\ldots dx_1$$
$$(\nu=0,1,\ldots,n-1),$$

we shall obtain on the basis of condition (2.2) the system of n-equations relative to the n-unknown $y^{(\nu)}(a)$.

In solving this system, we find

$$y^{(\nu)}(a)=e_{0\nu}+e_{1\nu}\int_a^b\varphi(x_1)\,dx_1+e_{2\nu}\int_a^b\int_a^{x_1}\varphi(x_2)\,dx_2\,dx_1+\ldots=$$

$$=\sum_{\rho=0}^{n} e_{\rho\nu}\underbrace{\int_a^b\int_a^{x_1}\ldots\int_a^{x_{\rho-1}}}_{\rho\ \text{times}}\varphi(x_\rho)\,dx_\rho\ldots dx_1 \qquad (2.11)$$
$$(\nu=0,1,\ldots,n-1).$$

The coefficients $e_{\rho\nu}$ are determined by coefficients, entering into boundary conditions (2.2).

By introducing, now the relationship (2.8) into equation (2.4), we obtain

$$\varphi=N\varphi+\sum_{k=0}^{n-1}\sum_{\rho=1}^{n}f_k(x)\,e_{\rho k}\underbrace{\int_a^b\ldots\int_a^{x_{\rho-1}}}_{\rho\ \text{times}}\varphi(x_\rho)\,dx_\rho\ldots dx_1+F, \qquad (2.12)$$

where

$$F=f(x)+\sum_{k=0}^{n-1}f_k(x)\,e_{0k}.$$

Equation (2.12) is a boundary integral equation

$$\varphi=K\varphi+F, \qquad (2.13)$$

where operator $K\varphi$ is expressed in first form [equality (1.2)]. The boundary integral equation (2.13) is equivalent to the differential equation (2.1) under boundary conditions of a general form.

We note also that the boundary integral operator is expressed in as the sum of

normal integral operator and linear series of functionals and given functions. This result can also be established directly from equality (1.2).

In practical problems of boundary conditions frequently they have a more simple structure. Suppose, for example, the value $y^{(k)}$ of the problem at $x=a_k$

$$y^{(k)}(a_k)=\gamma_k \qquad (k=0,1,\ldots,n-1), \tag{2.14}$$

the sections a_k usually coincide with ends of interval.

Boundary conditions of the form (2.14) we call simple. For obtaining a boundary integral equation it is sufficient in equalities (2.3) to select each time a lower limit of integration in such a way that condition (2.14), is satisfied.

For example:

$$y^{(n-1)}(x)=\gamma_{n-1}+\int_{a_{n-1}}^{x}\varphi(x_1)\,dx_1,$$

$$y^{(n-2)}(x)=\gamma_{n-2}+\gamma_{n-1}(x-a_{n-2})+\int_{a_{n-2}}^{x}\int_{a_{n-1}}^{x_1}\varphi(x_2)\,dx_2\,dx_1. \tag{2.15}$$

If conditions (2.14) for certain derivatives are not given, for example, for $y^{(\nu)}$, then the integration is made by assuming $a_\nu=a$, and initial value $y^{(\nu)}(a)$ is determined similarly to that, as was shown for a general case.

In certain cases, equation (2.1) is conveniently reduced to a matrix integral equation (1.17).

In making the integration in the intervals from a_{n-1} to x, we find

$$y^{(n-1)}(x)=-\int_{a_{n-1}}^{x}p_1(x_1)\,y^{(n-1)}(x_1)\,dx_1-\int_{a_{n-1}}^{x}p_2(x_1)\,y^{(n-2)}(x_1)\,dx_1-$$

$$-\int_{a_{n-1}}^{x}p_n(x_1)\,y(x_1)\,dx_1+\gamma_{n-1}+\int_{a_{n-1}}^{x}f(x_1)\,dx_1. \tag{2.16}$$

Furthermore one should consider the dependence

$$y^{(i)}(x)=\int_{a_i}^{x} y^{(i+1)}(x_1)\,dx_1+\gamma_i \qquad (i=0, 1, \ldots, n-2),$$

$$(\gamma_i=y^{(i)}(a_i)). \tag{2.17}$$

The system of equations (2.16) and (2.17) can be presented in matrix form:

$$\begin{bmatrix} y \\ y^{(1)} \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix}=\begin{bmatrix} 0 & K_{01} & 0 & & 0 \\ 0 & 0 & K_{12} & & 0 \\ & & & & \\ 0 & 0 & & 0 & K_{n-2,\,n-1} \\ K_{n-1,\,0} & K_{n-1,\,1} & & & K_{n-1,\,n-1} \end{bmatrix}\begin{bmatrix} y \\ y^{(1)} \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix}+$$

$$+\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_{n-2} \\ \gamma_{n-1} \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 0 \\ \int_{a_{n-1}}^{x} f(x_1)\,dx_1 \end{bmatrix}, \tag{2.18}$$

where

$$K_{i-1,\,i}\,y^{(i)}=\int_{a_{i-1}}^{x} y^{(i)}(x_1)\,dx_1$$

$$(i=1, 2, \ldots, n-1),$$

$$K_{n-1,\,i}\,y^{(i)}=-\int_{a_{n-1}}^{x} p_{n-i}(x_1)\,y^{(i)}(x_1)\,dx_1$$

$$(i=0, 1, \ldots, n-1).$$

In engineering problems frequently there are encountered binomial differential equations

$$\frac{d^{\nu_j}}{dx^{\nu_j}}\left\{p_j(x)\ldots\frac{d^{\nu_1}}{dx^{\nu_1}}\left[p_1(x)\frac{d^{\nu_0}}{dx^{\nu_0}}\left[p_0(x)\,y(x)\right]\right]\right\}-$$

$$-q(x)\,y(x)=f(x). \tag{2.19}$$

The formation of integral equation reduces in this case to a successive integration with proper selection of constants of the limits of integration.

Usually in practical problems

$$p_s(x) \neq 0, \quad s=0, 1, \ldots, j; \quad a \leqslant x \leqslant b;$$

in a converse case the coefficient in the prior derivative equation (2.12) vanishes at a certain point and the solution must contain a singular point.

Suppose, for example, there is given a differential equation for the vibration of a rod

$$\frac{d^2}{dx^2}\left[EJ(x)\frac{d^2y}{dx^2}\right]=p^2\rho F(x)\,y(x), \tag{2.20}$$

where y(x) is the amplitude sag of axis of rod;

EJ(x) is the strength of section to flexure;

ρ is the density of material of rod,

F(x) is the area of cross section,

p is the angular frequency of the natural oscillations.

We now consider a cantilever rod (Fig. 9), for which the boundary conditions have the form

$$y(0)=0, \quad y^{(1)}(0)=0, \quad y^{(2)}(l)=0, \quad \frac{d}{dx}\left(EJy^{(2)}(x)\right)\Big|_{x=l}=0. \tag{2.21}$$

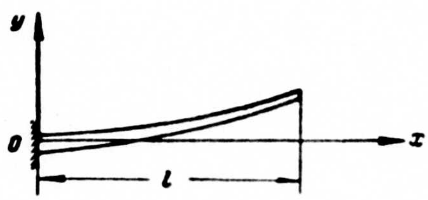

Fig. 9. Oscillations of rod.

By integrating both sides of equality (2.20) from x to l, we obtain

$$\frac{d}{dx}\left[EJ(x)\frac{d^2y}{dx^2}\right]=-p^2\int_x^l \rho F(x_1)\,y(x_1)\,dx_1.$$

By repeating the operation, we obtain

$$EJ(x)\frac{d^2y}{dx^2}=p^2\int_x^l\int_{x_1}^l \rho F(x_2)\,y(x_2)\,dx_2\,dx_1.$$

By transposing EJ(x) to the right side of equality and by integrating twice from 0 to x, we find

$$y(x)=p^2\int_0^x\int_0^{x_1}\frac{1}{EJ(x_2)}\int_{x_2}^l\int_{x_3}^l \rho F(x_4)\,y(x_4)\,dx_4\,dx_3\,dx_2\,dx_1. \tag{2.22}$$

Equation (2.22) is widely used in engineering computations, beginning with the works of P. F. Papkovich, E. P. Grossman and others. It is a homogeneous boundary integral equation.

There exist also other methods of formation integral equations from differential equations; they are reviewed later on in connection with applications.

3. The Solution of Homogeneous Boundary Integral Equations

Let us consider the solution of the equation

$$y = \lambda K y. \qquad (3.1)$$

Operator Ky is assumed to be symmetric, positively determined and the equation possesses the real eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ and the corresponding eigenfunctionals $y_1, y_2, y_3, \ldots$

The sequence of the eigenfunctions will form an orthonormal system in the interval $a<x<b$:

$$(y_i, y_j) = \int_a^b y_i(x)\, y_j(x)\, h(x)\, dx = \begin{cases} 1 & i=j, \\ 0 & i \neq j, \end{cases} \qquad (3.2)$$

where h(x) is a given positive function.

The indicated sequence is not, in general, complete, but if f(x) is an arbitrary function with a square being integrated, then on basis of theorem of the Gilbert--Schmidt function.

$$g = Kf \qquad (3.3)$$

is expanded into a uniformly and absolutely converging Fourier series

$$g(x) = \sum_{i=1}^{\infty} c_i y_i, \qquad (3.4)$$

where

$$c_i = (g, y_i).$$

Condition (3.3), in essence, denotes that the function g(x) may be an arbitrary boundary continuous function, satisfying the boundary conditions of the problem.

For solution of equation (3.1) there can be used with necessary changes, methods of solving homogeneous Fredholm integral equations. The most effective in

practical problems is found to be in most cases the method of successive approximations. For determining the first (minimum) eigenvalue and first eigenfunction the calculation is made according to the scheme

$$y_{(i)} = \lambda_{(i)} K y_{(i-1)}, \tag{3.5}$$

where $\lambda_{(i)}$ and $y_{(i)}$ are the i-th approximation for eigenvalue and eigenfunction.

The magnitude $\lambda_{(i)}$ is determined from condition of the very best "proximity" of initial and subsequent approximation.

In equating norms of the functions

$$\|y_{(i)}\| = \|y_{(i-1)}\|, \tag{3.6}$$

we obtain from equality (3.5)

$$\lambda_{(i)} = \frac{\|y_{(i-1)}\|}{\|K y_{(i-1)}\|}. \tag{3.7}$$

More accurate results (for a given approximation) are given with the use of a scalar norm of function

$$\|g\| = \sqrt{\int_a^b g^2 h\, dx}, \tag{3.8}$$

but more simple are the calculations peculiar to determining the norm of the function on basis of the maximum

$$\|g\| = \max_{a<x<b} |g|. \tag{3.9}$$

If we present the initial approximation $y_{(0)}$ expanded into a series according to the eigenfunctions

$$y_{(0)} = \sum_{n=1}^{\infty} c_n y_n, \tag{3.10}$$

then with the application of scalar norm we shall have

$$\lambda_{(1)} = \lambda_1 \sqrt{\frac{1+\left(\frac{c_2}{c_1}\right)^2 + \dots + \left(\frac{c_n}{c_1}\right)^2 + \dots}{1+\left(\frac{c_2}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_2}\right)^2 + \dots + \left(\frac{c_n}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_n}\right)^2 + \dots}},$$

$$\lambda_{(2)} = \lambda_1 \sqrt{\frac{1+\left(\frac{c_2}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_2}\right)^2 + \dots + \left(\frac{c_n}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_n}\right)^2 + \dots}{1+\left(\frac{c_2}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_2}\right)^4 + \dots + \left(\frac{c_n}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_n}\right)^4 + \dots}}. \tag{3.11}$$

At $i \to \infty$ $\lambda_{(i)} \to \lambda_1$, where approximations give for λ, an evaluation from above.

With the use of norm on basis of maximum we obtain

$$\lambda_{(1)} = \lambda_1 \frac{\left|1 + \frac{c_2}{c_1}\frac{y_2}{y_1} + \ldots + \frac{c_n}{c_1}\frac{y_n}{y_1} + \ldots\right|_{x=x_{m0}}}{\left|1 + \frac{c_2}{c_1}\frac{y_2}{y_1}\frac{\lambda_1}{\lambda_2} + \ldots + \frac{c_n}{c_1}\frac{y_n}{y_1}\frac{\lambda_1}{\lambda_n} + \ldots\right|_{x=x_{m0}}},$$

$$\lambda_{(2)} = \lambda_1 \frac{\left|1 + \frac{c_2}{c_1}\frac{y_2}{y_1}\frac{\lambda_1}{\lambda_2} + \ldots + \frac{c_n}{c_1}\frac{y_n}{y_1}\frac{\lambda_1}{\lambda_n}\right|_{x=x_{m1}}}{\left|1 + \frac{c_2}{c_1}\frac{y_2}{y_1}\left(\frac{\lambda_1}{\lambda_2}\right)^2 + \ldots + \frac{c_n}{c_1}\frac{y_n}{y_1}\left(\frac{\lambda_1}{\lambda_n}\right)^2 + \ldots\right|_{x=x_{m1}}}. \quad (3.12)$$

In these equalities, x_{m0} signifies the abscissa of section, corresponding to the maximum $|y_{(0)}|$. x_{m1} is the same for the first approximation $y_{(1)}$ et cetera.

If function y_1 does not vanish at one of the points $x_{m0}, x_{m1}, \ldots$ et cetera, then $\lambda_{(i)} \to \lambda_1$ with an increase of i.

In practical calculations the indicated limitation is immaterial, since the point x_{mi} tend to the point, where $y_{(i)}$ has a maximum value, and the first approximation always can be selected in such a manner that $y_1(x_{m0}) \neq 0$. This method may be called the method of comparing ordinates. The values $\lambda_{(i)}$ may be larger or smaller than λ_1 depending for example, on the selection of the initial approximation.

During the calculation it is convenient to assume

$$\|y_{(0)}\| = 1.$$

Then by virtue of equalities (3.6) and (3.7)

$$\lambda_{(i)} = \frac{1}{\|Ky_{(i-1)}\|}. \quad (3.13)$$

Suppose, for example, there is determined the frequency of flexural vibrations of a rod and equation (2.22) is used.

In selecting initial approximation in the form

$$y_{(0)} = \frac{x^2}{l^2}.$$

we obtain with the application of norm on basis of maximum

$$\omega^2_{(1)} = \frac{1}{\int\limits_0^l\int\limits_0^{x_1} \frac{1}{EJ(x_2)} \int\limits_{x_2}^l\int\limits_{x_3}^l pF(x_4)\frac{x_4^2}{l^2}\,dx_4\,dx_3\,dx_2\,dx_1}.$$

Usually $\omega_{(1)}$, determinate from this formula, differs from the accurate value by 3 to 5%.

With the use of the method of successive approximations [equality (3.5)] the eigenvalue can be found from condition of the minimum of square deviation with a "weight" h(x):

$$\varepsilon = \int\limits_a^b (y_{(i)} - y_{(i-1)})^2\, h\, dx = \int\limits_a^b (\lambda_{(i)} K y_{(i-1)} - y_{(i-1)})^2 h\, dx.$$

From the condition

$$\frac{\partial\varepsilon}{\partial\lambda_{(i)}} = 0$$

we obtain

$$\lambda_{(i)} = \frac{\int\limits_a^b y_{(i-1)} K y_{(i-1)}\, h\, dx}{\int\limits_a^b (K y_{(i-1)})^2 h\, dx}. \qquad (3.14)$$

In using equality (3.10), we find

$$\begin{aligned}
\lambda_{(1)} &= \lambda\,\frac{1+\left(\frac{c_2}{c_1}\right)^2\frac{\lambda_1}{\lambda_2}+\ldots+\left(\frac{c_n}{c_1}\right)^2\frac{\lambda_1}{\lambda_n}+\ldots}{1+\left(\frac{c_2}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_2}\right)^2+\ldots+\left(\frac{c_n}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_n}\right)^2+\ldots},\\
\lambda_{(2)} &= \lambda_1\,\frac{1+\left(\frac{c_2}{c_1}\right)^2\left(\frac{\lambda_2}{\lambda_1}\right)^3+\ldots+\left(\frac{c_n}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_n}\right)^3+\ldots}{1+\left(\frac{c_2}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_2}\right)^4+\ldots+\left(\frac{c_n}{c_1}\right)^2\left(\frac{\lambda_1}{\lambda_n}\right)^4+\ldots},
\end{aligned} \qquad (3.15)$$

whence it follows that $\lambda_{(i)} \to \lambda_1$ at $i \to \infty$, be giving always an evaluation from above.

A rapid convergence, peculiar to the above-indicated methods, is explained also by the fact that usually in equalities (3.12) and (3.15) coefficient c_1 significant larger than the remaining.

Let us turn to determining the second and highest eigenvalues.

It is possible as previously to proceed from the equation (3.1) in solving it by the method of successive approximations, but, as is known, the process of orthogonalization must be used both for the initial, and also for the subsequent approximation.

It is more convenient to use the transformed equation

$$y = \lambda K_1 y, \tag{3.16}$$

for which first eigenvalue is equal to the second eigenvalue of equation (3.1). In the theory of integral equations it appears that the equation possesses these properties

$$y(x) = \lambda \int_a^b \left(G(x,s) - \frac{y_1(x) y_1(s)}{\lambda_1} \right) h(s)\, y(s)\, ds$$

in relation to equation

$$y(x) = \lambda \int_a^b G(x,s)\, h(s)\, y(s)\, ds.$$

In application to boundary value integral equations we shall have

$$K_1 y = Ky - \frac{y_1}{\lambda_1} \frac{\int_a^b y(x) y_1(x) h(x)\, dx}{\int_a^b y_1^2(x) h(x)\, dx}. \tag{3.17}$$

However, this operator ensures the orthogonality for function y_1 only of the initial approximation and therefore it is not useful for practical calculations.

We shall indicate the structure of operator $K_2 y$, which in solving equation (3.16) by the method of successive approximations

$$y_{(i)} = \lambda_1 K_1 y_{(i-1)}$$

always assures orthogonality of a subsequent approximation $(y_{(i)})$ for the first eigenfunction independently of the selection of initial approximation.

This operator has the form

$$K_1 y = Ky - y_1 \frac{\int_a^b K y y_1 h\, dx}{\int_a^b y_1^2 h\, dx} \tag{3.18}$$

It is readily established

$$\int_a^b y_{(i)} y_1 h\,dx = \int_a^b K_1 y_{(i-1)} y_1 h\,dx = 0.$$

We note also that function y_1 is not assumed to be normalized.

Equation (3.16) is solved by the same methods as equation (3.1).

In determining the j-th eigenvalue (and eigenfunction) there is solved the equation

$$y = \lambda K_j y,$$

where

$$K_j y = Ky - \sum_{i=1}^{j-1} y_i \frac{\int_a^b K y y_i h\,dx}{\int_a^b y_i^2 h\,dx}.$$

We will now consider the solution by the method successive approximations of the matrix equation

$$[y] = \lambda\,[K][y]. \tag{3.19}$$

The eigen "functions" of this equation will be designated

$$[y_s] = \begin{bmatrix} y_{s,1} \\ y_{s,2} \\ \cdot \\ \cdot \\ y_{s,n} \end{bmatrix} \quad (s = 1, 2, 3, \ldots). \tag{3.20}$$

They satisfy condition of orthogonality (3.2) and normalization

$$([y_s], [y_r]) = \int_a^b \left(\sum_{j=1}^{n} y_{sj}(x)\, y_{rj}(x)\, h_j(x) \right) dx = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Equation (3.19) is solved on basis of the scheme

$$[y_{(i)}] = \lambda_{(i)}\,[K]\,[y_{(i-1)}],$$

where

$$[y_{(i-1)}] = \begin{bmatrix} y_{(i-1),1} \\ y_{(i-1),2} \\ \cdot \\ \cdot \\ \cdot \\ y_{(i-1),n} \end{bmatrix}, \quad [K][y_{(i-1)}] = \begin{bmatrix} \sum_{j=1}^{n} K_{1j} y_{(i-1),j} \\ \sum_{j=1}^{n} K_{2j} y_{(i-1),j} \\ \sum_{j=1}^{n} K_{nj} y_{(i-1),j} \end{bmatrix}.$$

In determining the i-th approximation for the first eigenvalue from the condition

$$\| [y_{(i)}] \| = \| [y_{(i-1)}] \|,$$

we obtain

$$\lambda_{(i)} = \frac{\| [y_{(i-1)}] \|}{\| [K][y_{(i-1)}] \|}.$$

In applying the norm on the basis of maximum (method of comparing ordinates)

$$\lambda_{(i)} = \left. \frac{\sqrt{y^2_{(i-1),1} + y^2_{(i-1),2} + \ldots + y^2_{(i-1),n}}}{\sqrt{\left(\sum_{j=1}^{n} K_{1j} y_{(i-1)j}\right)^2 + \ldots + \left(\sum_{j=1}^{n} K_{nj} y_{(i-1)j}\right)^2}} \right|_{x=x_{mi}} \tag{3.21}$$

or

$$\lambda_{(i)} = \left. \frac{\sqrt{\sum_{j=1}^{n} y^2_{(i-1),j}}}{\sqrt{\sum_{v=1}^{n} \left(\sum_{j=1}^{n} K_{vj} y_{(i-1),j}\right)^2}} \right|_{x=x_{mi}}, \tag{3.22}$$

where x_{mi} is the value $\underline{x}$, with which magnitude $\sqrt{\sum_{j=1}^{n} y^2_{(i-1),j}}$ has a maximum value.

With the application of the scalar norm

$$\lambda_{(i)} = \frac{\sqrt{\sum_{j=1}^{n} \int_a^b y^2_{(i-1),j} h_j \, dx}}{\sqrt{\sum_{v=1}^{n} \int_a^b \left(\sum_{j=1}^{n} K_{vj} y_{(i-1),j}\right)^2 h_v \, dx}}. \tag{3.23}$$

Less accurate results (for a given approximation) are given by a more simple method of determining $\lambda_{(i)}$, based on comparing one of the components $[y_{(i)}]$, for example, $y_{(i),r}$.

We shall have $y_{(i),r} = \lambda_{(i)} (K_{r1} y_{(i-1),1} + K_{r2} y_{(i-1),2} + \ldots + K_{rn} y_{(i-1),n})$

and further

$$\lambda_{(i)} = \left. \frac{y_{(i-1),r}}{\sum_{j=1}^{n} K_{rj} y_{(i-1),j}} \right|_{x=x_{mi}}. \tag{3.24}$$

where x_{mi} is the section, in which $|y_{(i-1),r}|$ has a maximum value.

It is possible to show that processes of solution by formulas [(3.22) -- (3.24)] converge to a minimum (in absolute value).

For determining the second eigenvalue, there is solved the equation

$$[y]=\lambda[K_2][y], \tag{3.25}$$

where
$$[K_2][y]=[K][y]-[y_1]\frac{([K][y],[y_1])}{\|[y_1]\|^2}.$$

In these equalities
$$[y]=\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \quad [y_1]=\begin{bmatrix} y_{11} \\ y_{12} \\ \cdot \\ \cdot \\ \cdot \\ y_{1n} \end{bmatrix}.$$

In a developed form, the matrix equation (3.25) is equivalent to the system of equations

$$\begin{aligned} y_1 &= \lambda\left(\sum_{j=1}^{n} K_{1j}y_j-\beta_1 y_{11}\right), \\ y_2 &= \lambda\left(\sum_{j=1}^{n} K_{2j}y_j-\beta_1 y_{12}\right), \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ y_n &= \lambda\left(\sum_{j=1}^{n} K_{nj}y_j-\beta_1 y_{1n}\right). \end{aligned} \tag{3.26}$$

Coefficient β_1 is equal to

$$\beta_1=\frac{([K][y],[y_1])}{\|[y_1]\|^2}=\frac{\int_a^b\left\{\sum_{\nu=1}^{n}\left(\sum_{j=1}^{n}K_{\nu j}y_j\right)y_{1\nu}h_\nu\right\}dx}{\int_a^b\left(\sum_{j=1}^{n}y_{1j}^2h_j\right)dx}.$$

Above-stated solution can be applied for calculating the flexure torsional vibrations of rods and in other problems.

Let us consider, now the solution of homogeneous integral equations (1.17):

$$\begin{bmatrix} y(x) \\ y^{(1)}(x) \\ \cdots \\ y^{(\nu)}(x) \end{bmatrix} = \lambda \begin{bmatrix} K_{00} & \cdots & K_{0\nu} \\ K_{10} & \cdots & K_{1\nu} \\ \cdots & \cdots & \cdots \\ K_{\nu 0} & \cdots & K_{\nu\nu} \end{bmatrix} \begin{bmatrix} y(x) \\ y^{(1)}(x) \\ \cdots \\ y^{(\nu)}(x) \end{bmatrix} \tag{3.27}$$

or more briefly written as

$$[y] = \lambda [K^{(\nu)}][y]. \tag{3.28}$$

We shall call a scalar, produce of an order of ν functions f and g with a "weight" $\underline{h}$, the following magnitude:

$$(f, g)^{(\nu)} = \int_a^b \sum_{j=1}^{\nu} f^{(j)} g^{(j)} h_j \, dx$$

In the considered case, the functions $h_j(x)$, $(j=0, \ldots, \nu)$ can be also negative.

We shall assume that matrix operator of equation (3.27) is symmetric:

$$([K^{(\nu)}][f], [g])^{(\nu)} = ([g], [K^{(\nu)}][f])^{(\nu)}. \tag{3.29}$$

For elastic systems this ensues from condition of reciprocity.

If operator $[Ky]$ is positively determinate,

$$([K^{(\nu)}][f], [f])^{(\nu)} > 0, \tag{3.30}$$

then equation (3.27) possesses real and positive eigenvalues.

Eigenfunctions of equation (3.27) y_i and y_j by virtue of conditions (3.29) are orthogonal and normalized

$$([y_i], [y_j])^{(\nu)} = \int_a^b \sum_{r=1}^{\nu} y_i^{(r)} y_j^{(r)} h_r \, dx = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Equation (3.28) is solved by the method of successive approximations:

$$[y_{(i)}] = \lambda_{(i)} [K^{(\nu)}][y_{(i-1)}]. \tag{3.31}$$

Let us consider first line of this equality:

$$y_{(i)} = \lambda_{(i)} K_0^{(\nu)} y_{(i-1)}, \tag{3.32}$$

where $$K_0^{(\nu)} y_{(i-1)} = K_{00} y_{(i-1)} + K_{01} y_{(i-1)}^{(1)} + \cdots + K_{0\nu} y_{(i-1)}^{(\nu)}.$$

By the method of comparing ordinates

$$y_{(i)} = y_{(i-1)}\big|_{x=x_{mi}}$$

we obtain

$$\lambda_{(i)} = \left.\frac{y_{(i-1)}}{K_0^{(\nu)} y_{(i-1)}}\right|_{x=x_{mi}}. \qquad (3.33)$$

The process thus constructed converges to the least eigenvalue in absolute value.

In determining the second eigenvalue the calculation is made by equation

$$[y] = \lambda\,[K_2^{(\nu)}][y], \qquad (3.34)$$

where

$$[K_2^{(\nu)}][y] = [K_1^{(\nu)}][y] - [y_1]\,\frac{([K^{(\nu)}][y],\,[y_1])^{(\nu)}}{\|[y_1]\|^2}.$$

In solving equation (3.34) previously indicated methods are used.

4. The Solution of Homogeneous Normal Integral Equations

The indicated equations have the form*

$$y = \lambda N y + \sum_{i=0}^{m} f_i L_i y. \qquad (4.1)$$

The number m, entering into this equation, corresponds to the number (of homogeneous) boundary conditions, which it must satisfy in the considered problem. It is necessary to remember that part of the boundary conditions (at x = a) is satisfied already in constructing the operator Ny.

Suppose, for example, there is considered the problem of flexure vibrations of a rod [equation (2.20)] with boundary conditions (2.21). In taking a constant limit of integration $a = l$, we obtain from equality (2.20) by successive integration

$$y(x) = \lambda \int_x^l \int_{x_1}^l \frac{1}{EJ(x_2)} \int_{x_2}^l \int_{x_3}^l \rho F(x_4)\, y(x_4)\, dx_4\, dx_3\, dx_2\, dx_1 + \\ + y(l)\, f_0(x) + y^{(1)}(l)\, f_1(x), \qquad (4.2)$$

*We recall that by our definition the equation is called homogeneous, if it admits a trivial solution, y = 0.

where

$$\lambda = a^2, \quad f_0(x) = 1, \quad f_1(x) = x - l. \tag{4.3}$$

Th[illegible] 'er two conditions (2.21) already were taken into account in the formation of equation (4.2).

Equation (4.2) is a normal integral equation.

$$L_0 y = y(0); \quad L_1 y = y^{(1)}(0)$$

The solution of equation (4.1) by virtue of the linearity of the integral operator is presented in the form

$$y = \sum_{i=0}^{m} L_i y \Phi_i(x), \tag{4.4}$$

where the function $\Phi_i(x)$ are expressed by absolutely and uniformly converging series

$$\Phi_i(x) = f_i + \lambda N f_i + \lambda^2 N^2 f_i + \ldots \quad (i = 0, 1, \ldots, m). \tag{4.5}$$

By introducing the relationship (4.4) into m+1 boundary value conditions, we shall obtain m+1 equations relative to the same number of unknown parameters $L_i y \ (i = 0, \ldots, m)$; by equating to zero, the determinant of the system, we find characteristic equation for determining the eigenvalues.

The indicated scheme in its basic features was used by Sh. E. Mikeladze.

We shall consider a practically important case, when equation (4.1) contains two functionals. This makes it possible to formulate also certain more general results. For definitiveness we shall consider equation (4.2).

The boundary value conditions at x = 0 (Fig. 9)

$$y(0) = 0; \quad y'(0) = 0$$

results in a system of equations

$$y(l)\Phi_0(0) + y^{(1)}(l)\Phi_1(0) = 0,$$
$$y(l)\Phi_0^{(1)}(0) + y^{(1)}(l)\Phi_1^{(1)}(0) = 0$$

and to the characteristic equation

$$\begin{vmatrix} \Phi_0(0) & \Phi_1(0) \\ \Phi_0^{(1)}(0) & \Phi_1^{(1)}(0) \end{vmatrix} = 0. \tag{4.6}$$

In determining the functions $\Phi_i(x)$ [equality (4.5)] will retain terms, corresponding to the "k-th approximation".

In this case

$$\Phi_0(0)=\sum_{\nu=0}^{k}a_\nu\lambda^\nu,\quad \Phi_1(0)=\sum_{\nu=0}^{k}d_\nu\lambda^\nu,$$
$$\Phi_0^{(1)}(0)=\sum_{\nu=0}^{k}c_\nu\lambda^\nu,\quad \Phi_1^{(1)}(0)=\sum_{\nu=0}^{k}b_\nu\lambda^\nu, \tag{4.7}$$

where

$$a_\nu=N^\nu f_0;\quad c_\nu=\frac{d}{dx}N^\nu f_0;\quad d_\nu=N^\nu f_1;\quad b_\nu=\frac{d}{dx}N^\nu f_1. \tag{4.8}$$

We note that for determining the derivatives $\Phi_i(x)$ special calculations are not required, since their values already are contained in calculation tables for determining $\Phi_i(x)$.

From equalities (4.6) and (4.7) it is evident

$$\sum_{\nu=0}^{k}a_\nu\lambda^\nu\sum_{\nu=0}^{k}b_\nu\lambda^\nu-\sum_{\nu=0}^{k}c_\nu\lambda^\nu\sum_{\nu=0}^{k}d_\nu\lambda^\nu=0 \tag{4.9}$$

or, by expanding into a series by degrees of λ, we obtain

$$F_k(\lambda)=0, \tag{4.10}$$

where the characteristic polynomial

$$F_k(\lambda)=\sum_{n=0}^{2k}A_{n(k)}\lambda^n, \tag{4.11}$$

$$A_{n(k)}=\begin{cases}\sum\limits_{i=0}^{n}(a_{n-i}b_i-c_{n-i}d_i) & 0\leqslant n\leqslant k,\\ \sum\limits_{i=0}^{2k-n}(a_{k-i}b_{n-k+i}-c_{k-i}d_{n-k+i}) & k+1\leqslant n\leqslant 2k.\end{cases} \tag{4.12}$$

In a limiting case at $k\to\infty$ we shall have

$$F(\lambda)=\sum_{n=0}^{\infty}A_n\lambda^n. \tag{4.13}$$

Expression (4.13) is a Fredholm series for the considered problem on eigenvalues.

The roots of the equation

$$F(\lambda)=0 \tag{4.14}$$

$\lambda_1,\lambda_2,\ldots,$ are all real and positive which ensues from symmetry and positive definitiveness of the corresponding boundary operator.

Let us assume, as usual, to number these roots (eigenvalues) in increasing order. Coefficients A_n with even n are positive, with odd, are negative; $A_0 = 1$.

From the equality

$$F(\lambda) = \prod_{i=1}^{\infty}\left(1-\frac{\lambda}{\lambda_i}\right) = \sum_{n=0}^{\infty} A_n \lambda^n$$

the known relationships follow

$$A_1 = -\sum_{i=1}^{\infty}\frac{1}{\lambda_i}, \qquad (4.15)$$

$$A_2 = \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{\lambda_i\lambda_j}$$

et cetera

The roots of the characteristic polynomial (4.14) are approximate values λ_i $(i=1, 2, \ldots)$ and possibly, in general, both real, and also conjugate complex.

There exists an important relationship, valid for equation with an arbitrary number of functionals; the characteristic polynomial of the k-th approximation has k+1 first coefficients, in accuracy conforming with corresponding coefficients of the Fredholm series.

Thus, for example, at k = 2

$$A_{0(2)} = A_0 = 1, \quad A_{1(2)} = A_1, \quad A_{2(2)} = A_2, \qquad (4.16)$$

but $A_{3(2)} \neq A_3$ et cetera.

With the use of following degrees of operator Ny, variations occur only in the terms with $n > k+1$.

This circumstance is important in practical problems, where it is usually required to determine several first eigenvalues, depending essentially only on first coefficients of the Fredholm series.

For seeking eigenvalues, i.e., roots of equation (4.14), it is expedient to apply Lobačevskiĭ method; by assuming a lack of multiple roots, we obtain for a first evaluation the approximate equalities

$$\lambda_{1(1)} \approx -\frac{1}{A_1},$$

$$\lambda_{2(1)} \approx -\frac{A_1}{A_2},$$

$$\lambda_{3(1)} \approx -\frac{A_2}{A_3}, \qquad (4.17)$$

$$\cdots\cdots$$

ensuing from relationship (4.15). For the second evaluation

$$\lambda_{1(2)} = \sqrt{-\frac{1}{A_1^{(2)}}},$$

$$\lambda_{2(2)} = \sqrt{-\frac{A_1^{(2)}}{A_2^{(2)}}},$$

$$\lambda_{3(2)} = \sqrt{-\frac{A_2^{(2)}}{A_3^{(2)}}},$$

$$\cdots\cdots \qquad (4.18)$$

Coefficients $A_i^{(2)}$ are coefficients of the expansion

$$F_{(2)}(\lambda) = F(\lambda)\,F(-\lambda) = \sum_{i=0}^{\infty} A_n^{(2)} (\lambda^2)^n.$$

These coefficients are determined by equality

$$A_n^{(2)} = \sum_{i=0}^{2n} A_{2n-i} A_i (-1)^i.$$

For seeking the accurate value $A_n^{(2)}$, there is required the use of the approximation of an order k = 2n. We note that equality (4.17) and (4.18) give in practical problems sufficient accuracy for the determination of the first two values.

It is possible to use in the calculation also some of the "inaccurate values" (following directly after accurate values in the characteristic polynomials) and then the matter reduces to seeking the first roots of the indicated polynomials.

Here, there may be used different methods, of which, in addition to the Lobačevskii method, we shall give attention to Newton's simple method and the method of graphic construction of function $F_k(\lambda)$.

In order to judge about accuracy of the calculation there should be made analogous calculations for the polynomial $F_{k-1}(\lambda)$.

It is possible to use upper and lower evaluations for the first and second eigenvalues*.

5. The Solution of Inhomogeneous Boundary Integral Equations

Let us consider methods of solving the equation

$$y = \lambda K y + f, \tag{5.1}$$

where λ is a given parameter; f is a given function $x (a < x < b)$.

Method of Simple Iteration

By applying usual process of successive approximations, we shall have

$$y_{(i)} = \lambda K y_{(i-1)} + f \qquad (i = 1, 2, 3, \ldots). \tag{5.2}$$

For the n-th approximation

$$y_{(n)} = f + \lambda K f + \lambda^2 K^2 f + \ldots + \lambda^{n-1} K^{n-1} f + \lambda^n K^n y_{(0)}, \tag{5.3}$$

where $K^i f$ is the i-th degree of the operator Kf:

$$K^i f = K (K^{i-1} f) \qquad (i = 1, 2, 3, \ldots). \tag{5.4}$$

The process of a simple iteration will be convergent, if the sequence of $y_{(n)}$ converges to an accurate solution

$$\lim_{n \to \infty} \{y_{(n)}\} = y.$$

We shall explain adequate conditions for the convergence of the series

$$y = f + \lambda K f + \lambda^2 K^2 f + \ldots = \sum_{i=0}^{\infty} \lambda^i K^i f. \tag{5.5}$$

Let us consider norm of the integral operator K. For a continuous operator

$$\|Kf\| < C \|f\|, \tag{5.6}$$

where

$$\|Kf\| = \max_{a<x<b} |Kf|, \qquad \|f\| = \max_{a<x<b} |f|.$$

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*S. A. Bernstein; Fundamentals of the Dynamics of Structures. Civil Engineer. Publ. House, Moscow, 1941; A. F. Smirnov, The Static and Dynamic Stability of Structures, Railway Transport Publ. House, Moscow, 1947.

The minimum value of the constant C, assuring inequality (5.6), is called the norm of the integral operator

$$C_{min} = \|K\|.$$

Thus,

$$\|Kf\| \leqslant \|K\| \, \|f\|. \tag{5.7}$$

In accordance with equality (1.2)

$$\|K\| \leqslant \max \left\{ |q_{011}| \left| \int_{a_{011}}^{x} |q_{111}| dx_1 \right| + |q_{012}| \left| \int_{a_{012}}^{x} |q_{112}| \int_{a_{112}}^{x_1} |q_{212}| dx_2 \, dx_1 \right| + \right.$$

$$\left. + |p_{011}| \left| \int_{a_{011}}^{b_{011}} |p_{111}| dx_1 \right| + |p_{012}| \left| \int_{a_{012}}^{b_{012}} |p_{112}| \int_{a_{112}}^{x_1} |p_{212}| dx_2 \, dx_1 \right| + \ldots \right\}. \tag{5.8}$$

Let us give an example.

Suppose

$$Kf = \int_0^x \int_1^{x_1} f(x_2) \, dx_2 \, dx_1, \qquad (0 \leqslant x \leqslant 1),$$

$$f(x) = -\frac{x}{3}.$$

Then

$$Kf = \frac{1}{6} x - \frac{x^3}{18};$$

$$|Kf| < \left| \int_0^x \int_1^{x_1} dx_2 \, dx_1 \right| \max_{0<x<1} |f(x)| \leqslant \left| \frac{x^2}{2} - x \right| \frac{1}{3};$$

$$\|Kf\| = \max_{0<x<1} |Kf| \leqslant \max_{0<x<1} \left| \frac{x^2}{2} - x \right| \frac{1}{3};$$

$$\|K\| \leqslant \max_{0<x<1} \left| \frac{x^2}{2} - x \right| = \frac{1}{2}.$$

In turning to a general case, we assume that

$$|\lambda| \cdot \|K\| = q < 1, \tag{5.9}$$

i.e.

$$|\lambda| < \frac{1}{\|K\|}. \tag{5.10}$$

In accordance with this

$$\max |\lambda^i K^i f| = \|\lambda^i K^i f\| \leqslant q^i \|f\|.$$

By virtue of the latter equalities the majorant series for y converges, in which

$$\|y\| < \frac{\|f\|}{1-|\lambda|\,\|K\|}. \tag{5.11}$$

This important result is a consequence of Banach's theorem* established for functional equations with linear operators.

Thus, under condition (5.9) the series (5.5) converges uniformly and absolutely; function, being expressed by the series (5.5), satisfies the integral equation (5.1), in which one may be convinced by direct substitution.

Further we shall establish that

$$\lim_{n\to\infty} \lambda^n K^n y_{(0)} = 0$$

for an arbitrary(bounded) function $y_{(0)}$ thus, the solution does not depend on selection of the initial approximation.

In connection with this, hencefore, we shall assume usually $y_{(0)} \equiv 0$ and then $y_{(1)} = f$.

We now turn to an evaluation of the error of successive approximations $y_{(n)}$:

$$\varepsilon_n = y - y_{(n)}, \tag{5.12}$$

where y is the accurate solution of equation (5.1).

From equality (5.3) at $y_{(0)} \equiv 0$ and expression (5.5) we obtain

$$\varepsilon_n = \lambda^n K^n f + \lambda^{n+1} K^n f + \ldots = \sum_{j=n}^{\infty} \lambda^j K^j f.$$

By virtue of (5.9) we shall have

$$\max_{a<x<b} |\varepsilon_n| = \|\varepsilon_n\| \leqslant \frac{\|f\|\, q^n}{1-q}. \tag{5.13}$$

It is of interest to establish also another evaluation for magnitude ε_n.

We shall designate difference between two successive approximations

$$\Delta y_{(n)} = y_{(n+1)} - y_{(n)}.$$

On the basis of equality (5.2)

$$\Delta y_{(n)} = \lambda K y_{(n)} + f - y_{(n)}; \tag{5.14}$$

the magnitude $\Delta y_{(n)}$ is the error in satisfying the main integral equation.

*L. V. Kantorovich. Functional Analysis and Applied Mathematics, "Advances in Mathematical Sciences", No. 6, 1948.

From relationships (5.12) and (5.14) we establish that error ε_n itself satisfies the integral equation

$$\varepsilon_n = \lambda K \varepsilon_n + \Delta y_{(n)}. \tag{5.15}$$

Now, by virtue of relationship (5.11)

$$\|\varepsilon_n\| \leqslant \frac{\|\Delta y_{(n)}\|}{1-|\lambda|\,\|K\|} = \frac{\|\Delta y_{(n)}\|}{1-q}, \tag{5.16}$$

which establishes connection between difference of two successive approximations and the error of the solution.

We consider now the practically important case, when the operator Ky has real eigenvalues $\lambda_i\ (i=1,2,\dots,\infty)$ and their corresponding eigenfunctions y_i.

If error of the initial approximation is presented in the form

$$\varepsilon_0 = \sum_{i=1}^{\infty} c_i y_i \tag{5.17}$$

and there is considered the relationship

$$\varepsilon_i = \lambda K \varepsilon_{i-1}, \tag{5.18}$$

ensuing from equalities (5.2) and (5.12), then we obtain

$$\varepsilon_n = \sum_{i=1}^{\infty} c_i \left(\frac{\lambda}{\lambda_i}\right)^n y_i. \tag{5.19}$$

From the latter equality, there ensues a well-known result: the process of simple iteration converges, if

$$|\lambda| < |\lambda_1|, \tag{5.20}$$

where λ_1 smallest (in absolute value) eigenvalue of the homogeneous integral equation

$$y = \lambda K y. \tag{5.21}$$

We note that in considered case (the homogeneous equation possesses infinite spectrum of eigenvalues) the solution of equation (5.1) may be obtained by the well-known expansion into series by eigenfunctions

$$y = \sum_{i=1}^{\infty} \frac{(f, y_i)}{1-\frac{\lambda}{\lambda_i}} y_i. \tag{5.22}$$

The use of this solution is very important for a theoretical analysis, practically it is less effective, than the application of method of successive approximations, since a determination of at least several first eigenfunctions is required.

Method of Complex Iteration

Let us now consider the case

$$|\lambda|>|\lambda_1|. \tag{5.23}$$

Of great practical value are equations, for which $\lambda<0$, but all the eigenvalues λ_i are positive (flexure of blades and turbomachine disks in the field of centrifugal forces, the flexure of beams on elastic foundation et cetera).

Thus,

$$y=-\mu Ky+f;\quad \mu>0; \tag{5.24}$$

$$y_i=\lambda_i Ky_i;\quad \lambda_i>0 \qquad (i=1, 2, 3, \ldots, \infty) \tag{5.25}$$

Process of complex iteration was shown by Viarda ("Integral equations") in connection with problem on longitudinally-transverse flexure of rods. According to this method

$$y_{(i+1)}=\alpha y_{(i)}+\beta(f-\mu Ky_{(i)}), \tag{5.26}$$

where α and β are parameters, identical for all approximations.

In accordance with equality (5.26) each subsequent approximation is a linear combination of the two preceding, obtained by the method of simple iteration.

Process can be generalized also for the case, when there is used a linear combination of several preceding approximations.

The error of the i-th approximation

$$\varepsilon_i=y-y_{(i)}=f-\mu Ky-\alpha y_{(i-1)}-\beta(f-\mu Ky_{(i-1)}) \tag{5.27}$$

can be presented in the following form:

$$\varepsilon_i=\alpha\varepsilon_{i-1}-\beta\mu K\varepsilon_{i-1}+(1-\alpha-\beta)y. \tag{5.28}$$

From this equality it follows that the necessary condition for

$$\varepsilon_n \underset{n\to\infty}{\to} 0 \tag{5.29}$$

will be such:

$$\alpha+\beta=1. \tag{5.30}$$

Now we obtain

$$e_i=\alpha e_{i-1}-\beta\mu K e_{i-1}, \quad (i=1,\ 2,\ \ldots\ \infty).$$

In presenting the error of initial approximation in the form (5.17), we find

$$e_n=\sum_{i=1}^{\infty} c_i\left(\alpha-\beta\frac{\mu}{\lambda_i}\right)^n y_i. \tag{5.31}$$

Equality (5.29) is found to be valid, if

$$\left|\alpha-\frac{\beta\mu}{\lambda_i}\right|=q_i<1,$$

or by taking into account the relationship (5.30)

$$\left|1-\beta\frac{\mu+\lambda_i}{\lambda_i}\right|=q_i<1 \quad (i=1,\ 2,\ 3,\ \ldots,\infty). \tag{5.32}$$

It remains to show that there can be found such a value β, that all inequalities (5.32) will be satisfied. This condition is essential, since at $\mu<0$ and $|\mu|>\lambda_1$ such a value is impossible to find.

It is possible to establish that inequality (5.32) at $\mu>0$ will be satisfied, if

$$0<\beta<\frac{2\lambda_1}{\mu+\lambda_1}, \tag{5.33}$$

where λ_1 is the minimum eigenvalue.

The value β expediently is selected with such a calculation that it corresponds with minimum values q_i, i.e., most rapid decrease of error. The solution of this question depends on relationships of the coefficients c_i, i.e., on the character of error, but also on the magnitude μ. If it is assumed that the coefficient of expansion of error by first form c_1 is considerably larger than the remaining, then one should select

$$\beta=\frac{\lambda_1}{\mu+\lambda_1}. \tag{5.34}$$

Then $q_1 = 0$, also at fairly large $i\,(\lambda_i\gg\mu)$

$$q_i=\frac{\mu}{\mu+\lambda_1}. \tag{5.35}$$

Viarda recommends the value

$$\beta=\frac{2\lambda_1}{\mu+2\lambda_1}, \tag{5.36}$$

with which for i = 1 also for fairly large i

$$q_i=\frac{\mu}{\mu+2\lambda_1}. \tag{5.37}$$

In accordance with equalities (5.26) and (5.30) the calculation is made according to equation

$$y_{(i+1)}=(1-\beta)\,y_{(i)}+\beta\,(f-\mu K y_{(i)}). \tag{5.38}$$

In assuming $y_{(0)}\equiv 0$, we find

$$y_{(1)}=\beta f$$

or, by using (5.34),

$$y_{(1)}=\frac{f}{1+\frac{\mu}{\lambda_1}}. \tag{5.39}$$

In solving the problem on longitudinally-transverse flexure of rods, we shall have

$$y_{(1)}=\frac{f}{1+\frac{N}{P_1}}, \tag{5.40}$$

where f – is the sag from effect of a transverse load; N-is tensile force, acting on the rod; P_1 -- critical force according to Euler. Formula (5.40), possessing great accuracy, is used in engineering calculations. It could have been obtained also from equality (5.22), if it is assumed that functions f and y_1 agree with an accuracy up to factor (curve of flexure from lateral load is similar to the first form for the loss of stability).

Process of complex iteration for the equation (5.24) is convergent at any values μ but at $\mu\gg\lambda_1$ the convergence is obtained more gradually since q_i is close to unity [See (5.35 and (5.37)].

Method of Complex Iteration With a Variable Parameter

A gradual convergence with large μ, in the preceding method is associated with the fact that parameter β was taken as constant for all approximations. Process of complex iteration [equality (5.38)] can be written out in the following form:

$$y_{(i+1)}=y_{(i)}+\beta\Delta y_{(i)}, \tag{5.41}$$

where

$$\Delta y_{(i)}=f-\mu K y_{(i)}-y_{(i)}. \tag{5.42}$$

Magnitude $\Delta y_{(i)}$ is the difference between two successive approximations or error in satisfying function $y_{(i)}$ of equation (5.24).

Process of complex iteration with variable parameter is expressed by the equality

$$y_{(i+1)}=y_{(i)}+\beta_{(i)}\Delta y_{(i)} \quad (i=0, 1, 2, \ldots), \tag{5.43}$$

where the parameter β_i can be determined from the condition, so that the function $y_{(i+1)}$ in the very best manner satisfies the main integral of equation (5.24).

For generality we shall return again to equation (5.1) and then

$$\Delta y_{(i)}=f+\lambda K y_{(i)}-y_{(i)}.$$

From the condition of minimum of the square deviation with the "weight" h(x) [the function h(x) enters into the condition of orthogonality of the eigenfunctions].

$$e_i=\int_a^b \Delta^2 y_{(i+1)}\, h\, dx, \tag{5.44}$$

where

$$\Delta y_{(i+1)}=f+\lambda K y_{(i+1)}-y_{(i+1)}, \tag{5.45}$$

we obtain

$$\frac{\partial e_i}{\partial \beta_i}=0. \tag{5.46}$$

From equalities (5.45) and (5.43) we find

$$\Delta y_{(i+1)}=\Delta y_{(i)}-\beta_i(\Delta y_{(i)}-\lambda K\Delta y_{(i)}) \tag{5.47}$$

and by virtue of (5.46)

$$\beta_i=\frac{\int_a^b \Delta y_{(i)}(\Delta y_{(i)}-\lambda K\Delta y_{(i)})\, h\, dx}{\int_a^b (\Delta y_{(i)}-\lambda K\Delta y_{(i)})^2\, h\, dx}. \tag{5.48}$$

It is possible to establish that

$$\frac{\partial^2 e_i}{\partial \beta_i^2}<0$$

and therefore conditions (5.46) determines minimum of the error.

We note that in determining β_i it is found necessary to calculate $K\Delta y_{(i)}$, which immediately is used in the following approximation:

$$y_{(i+2)} = y_{(i+1)} + \beta_{i+1}\Delta y_{(i+1)}.$$

The magnitude $\Delta y_{(i+1)}$ is determined from equality (5.47). The proof of the convergence of this method, as also of several subsequent methods is made difficult, however, it is obvious that if process of the solution converges, then it converges to an accurate solution.

The latter immediately ensues from equality (5.43), since, if

$$|y_{(i+1)} - y_{(i)}| < \delta,$$

then

$$|\Delta y_{(i)}| < \frac{\delta}{|\beta_i|}, \quad (\beta_i \neq 0)$$

and the function $y_{(i)}$ satisfies the integral equation with an error not exceeding $\frac{\delta}{|\beta_i|}$.

We shall show also that if function on the right side f can be expressed in form

$$f = c_i y_i,$$

then already the first approximation on basis of equality (5.43) results in an accurate solution (initial approximation is, as usual $y_{(0)} \equiv 0$).

We shall

$$y_{(1)} = \beta_1 f,$$

$$\beta_1 = \frac{\int_a^b f(f - \lambda K f)\,h\,dx}{\int_a^b (f - \lambda K f)^2\,h\,dx} = \frac{\int_a^b y_i\left(y_i - \frac{\lambda}{\lambda_i}y_i\right)h\,dx}{\int_a^b \left(y_i - \frac{\lambda}{\lambda_i}y_i\right)^2 h\,dx} = \frac{1}{1 - \frac{\lambda}{\lambda_i}}.$$

This result ensues also from equality (5.22).

We note that it is valid with the arbitrary function h. In practical calculations (for simplicity) it is possible to assume

$$h(x) = 1.$$

In problem about the flexure of turbomachine blades, propellers, the indicated process gives entirely satisfactory results at $\frac{|\lambda|}{\lambda_1}<10$.

Method of Similar Iteration

Let us assume that an approximation $y_{(i)}$ can be improved by multiplying be a certain coefficient $\beta_{(i)}$, i.e., can be assumed as the initial approximation

$$y^{*}_{(i)}=\beta_i y_{(i)}. \tag{5.49}$$

The subsequent approximation is determined from the equality

$$y_{(i+1)}=f+\beta_i\lambda K y_{(i)} \quad (i=0,\ 1,\ 2,\ 3,\ \ldots). \tag{5.50}$$

The coefficient β_i can be determined from different considerations. In assuming that the initial and subsequent approximations coincide at the point x_m $(a\leqslant x_m\leqslant b)$, where $y_{(i)}$ has a maximum value, we obtain

$$y_{(i+1)}(x_m)=y^{*}_{(i)}(x_m), \tag{5.51}$$

whence

$$\beta_i=\left.\frac{f}{y_{(i)}-\lambda K y_{(i)}}\right|_{x=x_m}.$$

Relationship (5.51) is equivalent to the condition of equality to zero of the error of integral equation at $x=x_m$, if into this equation is introduced "an improved" approximation:

$$\Delta y^{*}_{(i)}(x_m)=0.$$

The process of successive approximation constructed in this manner converges usually at $\left|\frac{\lambda}{\lambda_1}\right|<2$*. A similar method was used by V. P. Vetchinkin.

The significantly best results are obtained in the case, if β_i is determined from the condition

$$\int_a^b y_{(i+1)}\,dx=\int_a^b y^{*}_{(i)}\,dx\left(\text{or}\ \int_a^b \Delta y^{*}_{(i)}\,dx=0\right).$$

---

*In this section of instructions on convergence of processes are given on the basis of an experiment of their application during calculation of turbomachine blades.

which gives

$$\beta_i = \frac{\int_a^b f\,dx}{\int_a^b (y_{(i)} - \lambda K y_{(i)})\,dx}. \tag{5.52}$$

The process of solution, given by equalities (5.50) and (5.52), we shall call a similar iteration by the equality of areas. Practically this process converges at $\left|\frac{\lambda}{\lambda_1}\right| < 10$, where it is especially effective at $\left|\frac{\lambda}{\lambda_1}\right| < 5$.

The rapidity of convergence increases, if as an approximation of $y_{(i)}$ $(i > 3)$ there is taken the half sum of two preceding approximations:

$$y_{(i)} = \frac{1}{2}\left(y_{(i-1)} + y_{(i-2)}\right).$$

Finally, one can determine β_i and from the condition of the minimum of the square deviation of initial and subsequent approximation

$$e_i = \int_a^b (y_{(i+1)} - y^{\circ}_{(i)})^2\,dx \left(\text{or}\;\; e_i = \int_a^b \Delta y^{\circ 2}_{(i)}\,dx\right),$$

i.e. for equality

$$\frac{\partial e_i}{\partial \beta_i} = 0,$$

which gives

$$\beta_i = \frac{\int_a^b f(y_{(i)} - \lambda K y_{(i)})\,dx}{\int_a^b (y_{(i)} - \lambda K y_{(i)})^2\,dx}. \tag{5.53}$$

One of the important variants of the method of similar iteration (on basis of equality of functions) pointed out by S. A. Tumarkin*. Let the functions $y_{(i+1)}$ and $y^{\circ}_{(i)}$ coincide in all sections. This is possible at

$$\beta_i = \frac{f}{y_{(i)} - \lambda K y_{(i)}}. \tag{5.54}$$

If $y_{(i)}$ conforms with the accurate solution, then the parameter β_i will be constant; in reality, equality (5.54) determines the magnitude, depending on x.

In using in an approximate solution, equality (5.50) established for $\beta_i \neq \beta_i(x)$), we shall obtain

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*This method was communicated to author in a personal visit.

$$y_{(i+1)}=\frac{f}{y_{(i)}-\lambda K y_{(i)}}y_{(i)}$$
$$(i=0,\ 1,\ 2,\ \ldots) \tag{5.55}$$

The process of successive approximations converges in practical problems at $\left|\frac{\lambda}{\lambda_1}\right|<15$.

Relationship (5.55) loses meaning, when the denominator of the expression tends to zero which virtually is encountered fairly rarely.

We now shall point out that for all variants of method of similar iteration the following is valid:

A. If a process of subsequent approximations converges, then converges to an accurate solution:

B. If function f is similar to one of eigenfunctions, then already first approximation results in an accurate solution.

Suppose, therefore,

$$y_{(i+1)}\to y_{(i)}. \tag{5.56}$$

In order that $y_{(i+1)}$ tends toward an accurate solution, there must be in accordance with equality (5.50)

$$\beta_i\to 1. \tag{5.57}$$

Suppose, for example, the process of similar iteration is made on the basis of equality of areas:

$$y_{(i+1)}=\lambda K y_{(i)}\frac{\int_a^b f\,dx}{\int_a^b (y_{(i)}-\lambda K y_{(i)})\,dx}+f. \tag{5.58}$$

By integrating both sides of equality from a to b, we obtain

$$\int_a^b y_{(i+1)}\,dx=\int_a^b y_{(i)}\,dx\frac{\int_a^b f\,dx}{\int_a^b (y_{(i)}-\lambda K y_{(i)})\,dx}.$$

By virtue of (5.56)

$$\int_a^b y_{(i+1)}\,dx\to\int_a^b y_{(i)}\,dx,$$

whence also there ensues the relationship (5.57).

If the solution is constructed on basis of equation (5.55), then from (5.56) there ensues

$$y_{(i)} - \lambda K y_{(i)} \to f,$$

i.e., the function $y_{(i)}$ tends toward an accurate solution of the problem.

We shall now indicate the validity of the second assertion.

Suppose

$$f = c_n y_n,$$

where y_n is the n-th eigenfunction of operator Ky.

Then, by assuming, as usual, $y_{(0)} = f$, we obtain from equality (5.58)

$$y_{(1)} = f + \frac{\lambda}{\lambda_n} c_n y_n \frac{c_n \int_a^b y_n \, dx}{c_n \left(1 - \frac{\lambda}{\lambda_n}\right) \int_a^b y_n \, dx} = \frac{f}{1 - \frac{\lambda}{\lambda_n}}.$$

An analogous result is obtained from equality (5.55).

We note that "quality" of the approximation can be evaluated also by the magnitude β_i. With a good convergence of the process already for second or third approximation $\beta_i \approx 1$.

We shall dwell briefly on the solution of matrix equations

$$[y] = \lambda [Ky][y] + [f]. \qquad (5.59)$$

We shall assume that the corresponding homogeneous equation has the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$

The process of simple iteration for equation (5.59)

$$[y_{(i+1)}] = \lambda [K][y_{(i)}] + f$$

converges, if

$$|\lambda| < |\lambda_1|.$$

At $|\lambda| > |\lambda_1|$ there are constructed processes of iteration, analogous to the previously indicated processes.

We shall consider as example the application of the method of similar iteration.

In this case $$[y_{(i+1)}] = \beta_i \lambda [K][y_{(i)}] + [f].$$

The magnitude β_i is determined from following condition: the corrected approximation

$$[y^\circ_{(i)}] = \beta_i [y_{(i)}]$$

in the very best manner has to satisfy equation (5.59).

The error of equation (5.59)

$$[\Delta y^\circ_{(i)}] = \beta_i \lambda [K][y_{(i)}] + [f] - \beta_i [y_{(i)}]. \tag{5.60}$$

Thus,

$$[\Delta y^\circ_{(i)}] = \begin{bmatrix} \beta_i \lambda \sum\limits_{j=1}^{n} K_{1j} y_{(i),j} + f_1 - \beta_i y_{(i),1} \\ \beta_i \lambda \sum\limits_{j=1}^{n} K_{2j} y_{(i),j} + f_2 - \beta_i y_{(i),2} \\ \cdots\cdots\cdots\cdots \\ \beta_i \lambda \sum\limits_{j=1}^{n} K_{nj} y_{(i),n} + f_n - \beta_i y_{(i),n} \end{bmatrix}.$$

The absolute value of the error

$$|[\Delta y^\circ_{(i)}]| = \sqrt{\sum_{\nu=1}^{n} \left\{ \beta_i \left(\lambda \sum_{j=1}^{n} K_{\nu j} y_{(i),j} - y_{(i),\nu} \right) + f_\nu \right\}^2}.$$

By determining β_i from the condition of minimum of square deviation $e_i = \int\limits_a^b |[\Delta y^\circ_{(i)}]|^2 dx$, i.e. from the equality $\frac{\partial e_i}{\partial \beta_i} = 0$, we obtain

$$\beta_i = \frac{\sum\limits_{\nu=1}^{n} \int\limits_a^b f_\nu \left(y_{(i),\nu} - \lambda \sum\limits_{j=1}^{n} K_{\nu j} y_{(i),j} \right) dx}{\sum\limits_{\nu=1}^{n} \int\limits_a^b \left(y_{(i),\nu} - \lambda \sum\limits_{j=1}^{n} K_{\nu j} y_{(i),j} \right)^2 dx}.$$

If we take

$$[y^\circ_{(i)}] = \begin{bmatrix} \beta_{i1} y_{(i),1} \\ \cdots \\ \beta_{in} y_{(i),n} \end{bmatrix},$$

then for the method of similar iteration on the basis of the equality of functions we obtain the system of equations (at $f_\nu \neq 0$)

$$y_{(i+1),\nu} = \frac{f_\nu y_{(i),\nu}}{y_{(i),\nu} - \lambda \sum\limits_{j=1}^{n} K_{\nu j} y_{(i),j}}; \tag{5.61}$$

$$(\nu = 1, 2, \ldots, n).$$

There exist also different methods of approximation, reducing the problem to solution of a system of linear algebraic equations (Fredholm method, method of moments, method of collocation and others). In practical problems, these methods usually are inferior to the previously indicated methods.

6. The Solution of Inhomogeneous Normal Integral Equations

In the solution of equation (1.5) its component functionals are considered as parameters and for brevity are designated

$$L_i y = C_i. \tag{6.1}$$

In solving a normal equation by the method of successive approximations

$$y_{(i+1)} = \lambda N y_{(i)} + \sum_{i=0}^{m} C_i f_i + f \tag{6.2}$$

we obtain

$$y = \sum_{i=0}^{m} C_i \Phi_i + \Phi, \tag{6.3}$$

where

$$\Phi_i = f_i + \lambda N f_i + \lambda^2 N^2 f_i + \ldots, \tag{6.4}$$

$$\Phi = f + \lambda N f + \lambda^2 N^2 f + \ldots \tag{6.5}$$

The parameters C_i are determined from boundary conditions of the problem. Series (6.4) and (6.5) converge uniformly and absolutely with an arbitrary value λ. Let us note that if calculation is made on basis of equation (6.2) and the parameters C_i are determined after each approximation, then process converges only at

$$|\lambda| < |\lambda_1|.$$

In "deciphering" the C_i values in accordance with equality (6.1) the normal equation acquires all the properties of a boundary equation.

Sometimes it is convenient for the series (6.4) and (6.5) to be used in another form, for example,

$$\Phi_i = \Phi_{i(0)} + \Phi_{i(1)} + \Phi_{i(2)} + \ldots, \tag{6.6}$$

where

$$\Phi_{i(s)} = \lambda N \Phi_{i(s-1)} \quad (s = 1, 2, 3, \ldots), \quad (\Phi_{i(0)} = f_i).$$

In above mentioned series each term of the series is the difference between two successive approximations. The calculations are stopped, when a new term of the series is small in comparison with sum of all the obtained terms of the series. In evaluating the convergence of the process of successive approximations it can be established that the terms entering into the series (6.4)-(6.6) diminish with an accuracy up to a factor, as $\frac{[\lambda(b-a)]^n}{n!}$. For large values of the parameter λ and of the interval of determining the function, the convergence deteriorates and in a number of practical problems even at $\left|\frac{\lambda}{\lambda_1}\right|>5$ becomes gradual. For improving the convergence, it is possible to apply method of "mobile origin", with which the function is found at first in the fairly small section $a<x<a_1$, which assures a rapid convergence. Further, the solution is constructed for the following section (in normal integral equation it is assumed $a=a_1$) and initial parameters in section $x=a_1$ are determined from the preceding solution. The condition of "rapid convergence" of the simple iteration process for a normal equation can be recommended in such a form:

$$|\lambda|\|N\|<1. \tag{6.7}$$

In a number of problems in structural mechanics under condition (6.7) there is required not more than three to four approximations.

Analogous results are obtained in solving matrix equations

$$[y]=\lambda[N][y]+\sum_{i=0}^{m}C_i[f_i]+[f]$$

by the method of successive approximations; we shall

$$[y]=\sum_{i=0}^{m}C_i[\Phi_i]+[\Phi],$$

$$[\Phi_i]=[f_i]+\lambda[N][f_i]+\lambda^2[N]^2[f_i]+\ldots.$$

$$[\Phi]=[f]+\lambda[N][f]+\lambda^2[N]^2[f]+\ldots$$

the functions $[\Phi_i]$ and $[\Phi]$ can be presented in the form of a series, for example,

$$[\Phi]=\sum_{j=0}^{m}[\Phi_{(j)}].$$

in which

$$[\Phi_{(j)}]=\lambda[N][\Phi_{(j-1)}], \quad [\Phi_{(0)}]=[f].$$

We shall now consider the application of the linear approximation method, in which there is used an approximate integration by the trapezoidal rule.

The indicated method for boundary integral equations results in the necessity of solving a system of algebraic equations; for normal integral equations the matter is greatly simplified because the values $y(x_j)$ $(j=0, 1, 2, \ldots)$ can be determined successively one after another.

Suppose the normal equation is given in form

$$y(x)=\sum_{s=1}^{n} Q_s(x)\int_a^x q_s(x_1)\,y(x_1)\,dx_1+f+\sum_{i=0}^{m} C_i f_i, \tag{6.8}$$

[parameter λ is contained in the function $Q_s(x)$].

Solution of equation (6.8) will have form (6.3), and, for example, for determining function $\Phi(x)$ there must be solved the equation

$$y(x)=\sum_{s=1}^{n} Q_s(x)\int_a^x q_s(x_1)\,y(x_1)\,dx_1+f. \tag{6.9}$$

Now we consider first variant of method of linear approximation. We shall subdivide the interval of x variation into k sectors with the sections $x_0=a$, $x_1, \ldots x_j, \ldots, x_k=b$.

The length of a sector

$$x_j-x_{j-1}=\Delta_j.$$

we shall designate

$$y(x_j)=y_j, \quad Q_s(x_j)=Q_{sj}, \quad f(x_j)=f_j,$$
$$j=0, 1, 2, \ldots, k,$$
$$s=1, \ldots, n.$$

On the basis of equality (6.9) we shall establish

$$y_j=\frac{1}{2}\sum_{s=1}^{n} Q_{sj}\left[q_{s0}y_0\Delta_1+\sum_{\nu=1}^{j-1} q_{s\nu}y_\nu(\Delta_\nu+\Delta_{\nu+1})+q_{sj}y_j\Delta_j\right]+f_j, \tag{6.10}$$
$$(j=1, 2, \ldots, k, \quad y_0=f_0).$$

Hence,

$$y_j = \frac{1}{1 - \frac{1}{2}\sum_{s=1}^{n} Q_j q_{sj} \Delta_j} \times$$

$$\times \left\{ \frac{1}{2} \sum_{s=1}^{n} Q_{sj} \left[q_{s0} y_0 \Delta_1 + \sum_{\nu=1}^{j-1} q_{s\nu} y_\nu (\Delta_\nu + \Delta_{\nu+1}) \right] + f_j \right\}.$$

In final form

$$y_j = \frac{1}{1 - a_{jj}} \left(\sum_{\nu=0}^{j-1} a_{j\nu} y_\nu + f_j \right) \qquad (6.11)$$

$$(j = 1, 2, \ldots, k),$$

where

$$a_{j\nu} = \frac{1}{2} \sum_{s=1}^{n} Q_{sj} q_{s\nu} (\Delta_\nu + \Delta_{\nu+1})$$

$$(j = 1, 2, \ldots, k; \quad \nu = 1, 2, \ldots, j-1),$$

$$a_{j0} = \frac{1}{2} \sum_{s=1}^{n} Q_{sj} q_{s0} \Delta_1,$$

$$a_{jj} = \frac{1}{2} \sum_{s=1}^{n} Q_{sj} q_{sj} \Delta_j.$$

Calculation by equation (6.11) is convenient to make according to scheme, shown in Table 5. Magnitude $y_0 = f_0$ is multiplied by the column "0" of the table $a_{j\nu}$ and the result is filled in column "0" of table $a_{j\nu} y_\nu$. By summarizing the numbers in first line and by dividing by $1 - a_{11}$, we find y_1. By multiplying y_1 by column "1" of table $a_{j\nu}$ et cetera, we find all the values y_j.

A simplification of the calculation is attained by subdividing the section into equal intervals.

With equal intervals, in addition to a linear approximation (integrating by trapezodial rule), there may be used integration by Simpson's rule et cetera which

Table 5. Calculation Scheme According to Linear Approximation Method

		$a_{j\nu}$					$a_{j\nu}y_\nu$							
x	$j \backslash \nu$	0	1	2	...	k	0	1	2	...	k	f	$\Sigma a_{j\nu}y_\nu+f_j$	y
x_0	0				...					...		f_0	f_0	$y_0=f_0$
x_1	1	a_{10}	a_{11}		...		$a_{10}y_0$			...		f_1	$a_{10}y_0+f_1$	y_1
x_2	2	a_{20}	a_{21}	a_{22}	...		$a_{20}y_0$	$a_{21}y_1$		...		f_2		y_2
					...					...				
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
x_k	k	a_{k0}	a_{k1}	a_{k2}	...	a_{kk}	$a_{k0}y_0$	$a_{k1}y_1$	$a_{k2}y_2$	...		f_k		y_k

results in the appearance of corresponding factors for the coefficients q_{sj}.
The interpolation formulas by Chebyshev and Gauss are not successfully applied in a similar manner, since for different sectors the points of interpolation will not be common.

We now discuss the second variant of the linear approximation which leads to another scheme of calculation. We shall write equality (6.10) in such a form:

$$y_j = \sum_{s=1}^{n} Q_{sj}\left(J_{s,j-1} + \frac{1}{2} q_{sj} y_j \Delta_j\right) + f_j, \tag{6.12}$$

where

$$J_{s,j} = J_{s,j-1} + \frac{1}{2} q_{sj} y_j (\Delta_j + \Delta_{j+1}), \quad (j = 1, 2, \ldots, k-1), \quad \left(J_{s,0} = \frac{1}{2} q_{s0} y_0 \Delta_1\right). \tag{6.13}$$

From relationship (6.12) we shall obtain the calculation formula

$$y_j = \frac{1}{1 - A_j}\left(\sum_{s=1}^{n} Q_{sj} J_{s,j-1} + f_j\right), \quad (j = 1, 2, \ldots, k); \quad (y_0 = f_0), \quad A_j = \frac{1}{2} \sum_{s=1}^{n} Q_{sj} q_{sj} \Delta_j. \tag{6.14}$$

Thus,

$$y_0 = f_0,$$

$$y_1 = \frac{1}{1 - \frac{1}{2}\sum_{s=1}^{n} Q_{s1} q_{s1} \Delta_1}\left(\sum_{s=1}^{n} Q_{s1} J_{s0} + f_1\right) \quad \left(J_{s0} = \frac{1}{2} q_{s0} y_0 \Delta_1\right),$$

$$y_2 = \frac{1}{1 - \frac{1}{2}\sum_{s=1}^{n} Q_{s2} q_{s2} \Delta_2}\left(\sum_{s=1}^{n} Q_{s2} J_{s1} + f_2\right) \quad \left(J_{s1} = J_{s0} + \frac{1}{2} q_{s1} y_1 (\Delta_1 + \Delta_2)\right), \ldots$$

With a small number of calculation sections $(k<8)$ the more convenient is the first variant of linear approximation method, with a large number of sections, the second variant.

We now consider the application of the linear approximation method for the solution of matrix normal integral equations

$$[y]=\lambda[N][y]+[f].$$

As an example we take the equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The magnitude of the known parameter λ is included in the operators.

If normal operators are given in the second form [equation (6.8)], then in an expanded writing we shall have

$$\begin{aligned} y_1(x) &= \sum_{s=1}^{m_1} Q_{1s}(x)\int_a^x q_{1s}(x_1)\,y_1(x_1)\,dx_1 + \\ &+ \sum_{s=1}^{m_1} T_{1s}(x)\int_a^x t_{1s}(x_1)\,y_2(x_1)\,dx_1 + f_1. \\ y_2(x) &= \sum_{s=1}^{m_2} Q_{2s}(x)\int_a^x q_{2s}(x_1)\,y_1(x_1)\,dx_1 + \\ &+ \sum_{s=1}^{m_2} T_{2s}(x)\int_a^x t_{2s}(x_1)\,y_2(x_1)\,dx_1 + f_2. \end{aligned} \qquad (6.15)$$

As earlier, we shall briefly designate

$$y_1(x_j)=y_{1j};\quad Q_{1s}(x_j)=Q_{1sj};\quad q_{1s}(x_j)=q_{1sj};\ \ldots \quad \text{et cetera}$$

In using the second variant of the method, similar to relationship (6.12) we obtain

$$\begin{aligned} y_{1j} &= \sum_{s=1}^{m_1} Q_{1sj}\left(J^{(q)}_{1s,\,j-1}+\frac{1}{2}\,q_{1sj}y_{1j}\Delta_j\right)+ \\ &+ \sum_{s=1}^{m_1} T_{1sj}\left(J^{(t)}_{1s,\,j-1}+\frac{1}{2}\,t_{1sj}y_{2j}\Delta_j\right)+f_{1j}, \\ y_{2j} &= \sum_{s=1}^{m_2} Q_{2sj}\left(J^{(q)}_{2s,\,j-1}+\frac{1}{2}\,q_{2sj}y_{1j}\Delta_j\right)+ \\ &+ \sum_{s=1}^{m_2} T_{2sj}\left(J^{(t)}_{2s,\,j-1}+\frac{1}{2}\,t_{2sj}y_{2j}\Delta_j\right)+f_{2j}. \end{aligned} \qquad (6.16)$$

where

$$J^{(q)}_{1s,j} = J^{(q)}_{1s,j-1} + \frac{1}{2} q_{1sj} y_j (\Delta_j + \Delta_{j+1})$$

$$(j = 1, 2, \ldots, k-1),$$

$$J^{(q)}_{1s,0} = \frac{1}{2} q_{1s,0} y_0 \Delta_1$$

and t. p.

Equality (6.16) will be written as:

$$\begin{aligned} a_{11j} y_{1j} + a_{12j} y_{2j} &= F_{1j}, \\ a_{21j} y_{1j} + a_{22j} y_{2j} &= F_{2j}, \end{aligned} \tag{6.17}$$

where

$$a_{11j} = 1 - \frac{1}{2} \Delta_j \sum_{s=1}^{n} Q_{1sj} q_{1sj},$$

$$a_{12j} = -\frac{1}{2} \Delta_j \sum_{s=1}^{m_1} T_{1sj} t_{1sj},$$

$$a_{21j} = -\frac{1}{2} \Delta_j \sum_{s=1}^{n} Q_{2sj} q_{2sj},$$

$$a_{22j} = 1 - \frac{1}{2} \Delta_j \sum_{s=1}^{m_2} T_{2sj} t_{2sj},$$

$$F_{1j} = f_{1j} + \sum_{s=1}^{n_1} Q_{sj} J^{(q)}_{1s,j-1} + \sum_{s=1}^{m_1} T_{1sj} J^{(t)}_{1s,j-1},$$

$$F_{2j} = f_{2j} + \sum_{s=1}^{n_2} Q_{2sj} J^{(q)}_{2s,j-1} + \sum_{s=1}^{m_2} T_{2sj} J^{(t)}_{2s,j-1}.$$

From equations (6.17) we obtain the calculation formulas

$$y_{1j} = \frac{a_{22j} F_{1j} - a_{12j} F_{2j}}{a_{11j} a_{22j} - a_{12j} a_{21j}},$$

$$y_{2j} = \frac{a_{11j} F_{2j} - a_{21j} F_{1j}}{a_{11j} a_{22j} - a_{12j} a_{21j}}$$

$$(j = 1, 2, \ldots, k).$$

As previously, the values $y_1(x)$ and $y_2(x)$ in each calculation are determined in sequence.

CHAPTER 4

APPLICATION OF BOUNDARY AND NORMAL INTEGRAL EQUATIONS TO PROBLEMS OF STRUCTURAL MECHANICS

1. Flexure of Rods in a Field of Centrifugal Forces

This problem has a number of engineering applications in calculating for the strength of blades of steam and gas turbines, compressors, blades of propellers and helicopters. The application of method of successive approximations and boundary integral equations in calculating for the strength of propellers is given in works by V. P. Vetchinkin, D. Yu. Panov, P. M. Riza, S. A. Tumarkin.

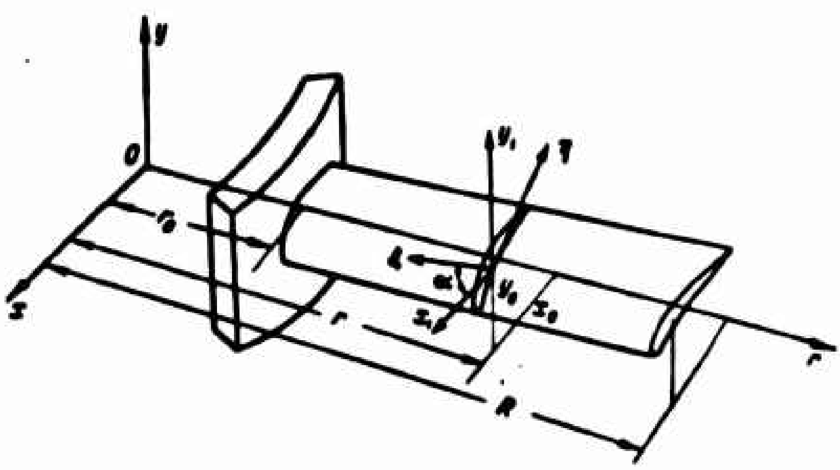

Fig. 10. Flexure of rod in a field of centrifugal forces.

In Fig. 10 there is shown the system of coordinates being used. The y axis coincides with axis of rotation, the r axis is directed radially and passes

through the center of gravity of the root section of rod. Axis of rod is assumed in the form of space curve, deviating little from a radial direction.

The origin of the local axes $x_1 y_1$ is placed at the center of gravity of cross section; principal axes of the section ξ, η are turned by an angle α in reference to the local system.

Equations of the flexure of a naturally twisted rod have the form:*

$$\frac{d^2u}{dr^2} = \left(\frac{\cos^2\alpha}{EJ_\eta} + \frac{\sin^2\alpha}{EJ_\xi}\right) M_{y1} - \frac{1}{2}\left(\frac{1}{EJ_\eta} - \frac{1}{EJ_\xi}\right) \sin 2\alpha\, M_{x1},$$
$$\frac{d^2v}{dr^2} = -\left(\frac{\sin^2\alpha}{EJ_\eta} + \frac{\cos^2\alpha}{EJ_\xi}\right) M_{x1} + \frac{1}{2}\left(\frac{1}{EJ_\eta} - \frac{1}{EJ_\xi}\right) \sin 2\alpha\, M_{y1}, \tag{1.1}$$

where u and v are the elastic displacements of axis of rod along the x and y axes respectively; EJ_η and EJ_ξ --are the primary rigidities of section during flexure. The bending moments M_{x1} and M_{y1} during flexure of rod in a field of centrifugal forces are equal to

$$M_{x1}(r) = \tilde{M}_{x1}(r) + \rho\omega^2 \int_r^R (v(r_1) - v(r))\, r_1 F(r_1)\, dr_1,$$
$$M_{y1}(r) = \tilde{M}_{y1}(r) + \rho\omega^2 \int_r^R (r_1 u(r) - r u(r_1))\, F(r_1)\, dr_1, \tag{1.2}$$

where ρ is the density of the material; ω is the angular velocity; F -- across section area. $\tilde{M}_{x1}(r)$ and $\tilde{M}_{y1}(r)$ designate bending moments from a transverse load p_x and p_y and the initial distortion of axis [its coordinates prior to deformation $x_0(r)$ $y_0(r)$]:

$$\tilde{M}_{x1}(r) = -\int_r^R \int_{r_1}^R p_y(r_2)\, dr_2\, dr_1 + \rho\omega^2 \int_r^R (y_0(r_1) - y_0(r))\, r_1 F(r_1)\, dr_1,$$
$$\tilde{M}_{y1}(r) = \int_r^R \int_{r_1}^R p_x(r_2)\, dr_2\, dr_1 + \rho\omega^2 \int_r^R (r_1 x_0(r) - r x_0(r_1))\, F(r_1)\, dr_1. \tag{1.3}$$

Equation (1.1) which takes into account relationships (1.2) will form a system of integro-differential equations. We shall transform it into a matrix boundary integral equation.

As the chief unknowns we shall take the components of the curvature

$$\frac{d^2u}{dr^2} = \varphi(r),$$
$$\frac{d^2v}{dr^2} = \psi(r). \tag{1.4}$$

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*S. A. Tumarkin, Equilibrium and Oscillation of Twisted Rods, Transactions of the Central Aero-Hydrodynamic Institute, No. 341, 1937.

By means of integration by parts from relationships (1.2) we find

$$M_{x1}(r) = \tilde{M}_{x1}(r) + \omega^2 \int_r^R C_1(r_1) \frac{dv}{dr_1}(r_1)\, dr_1,$$
$$M_{y1}(r) = \tilde{M}_{y1}(r) - \omega^2 r \int_r^R C_1(r_1) \frac{d}{dr_1}\left(\frac{u(r_1)}{r_1}\right) dr_1, \tag{1.5}$$

where

$$C_1(r) = \rho \int_r^R r_1 F(r_1)\, dr_1 = \frac{C(r)}{\omega^2};$$

C(r) is the centrifugal force, acting in the section r. For a rigidly fixed blade in the root section

$$\frac{d}{dr}\left(\frac{u(r)}{r}\right) = \frac{1}{r^2} \int_{r_0}^{r} r_1 \varphi(r_1)\, dr_1,$$
$$\frac{dv}{dr} = \int_{r_0}^{r} \psi(r_1)\, dr_1. \tag{1.6}$$

By introducing equality (1.5) with a consideration of the dependence (1.6) in equation (1.1), we obtain

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{bmatrix} \tag{1.7}$$

where the parameter of equation

$$\lambda = -\omega^2, \tag{1.8}$$

and integral operators are determined by the equalities

$$K_{11}\varphi = \left(\frac{\cos^2 \alpha}{EJ_\eta} + \frac{\sin^2 \alpha}{EJ_\xi}\right) r \int_r^R C_1(r_1) \frac{1}{r_1^2} \int_{r_0}^{r_1} r_2 \varphi(r_2)\, dr_2\, dr_1;$$
$$K_{12}\psi = \frac{1}{2}\left(\frac{1}{EJ_\eta} - \frac{1}{EJ_\xi}\right) \sin 2\alpha \int_r^R C_1(r_1) \int_{r_0}^{r_1} \psi(r_2)\, dr_2\, dr_1;$$
$$K_{21}\varphi = \frac{1}{2}\left(\frac{1}{EJ_\eta} - \frac{1}{EJ_\xi}\right) \sin 2\alpha \cdot r \int_r^R C_1(r_1) \frac{1}{r_1^2} \int_{r_0}^{r_1} r_2 \varphi(r_2)\, dr_2\, dr_1; \tag{1.9}$$
$$K_{22}\psi = \left(\frac{\sin^2 \alpha}{EJ_\eta} + \frac{\cos^2 \alpha}{EJ_\xi}\right) \int_r^R C_1(r_1) \int_{r_0}^{r_1} \psi(r_2)\, dr_2\, dr_1.$$

The magnitudes $\tilde{\varphi}$ and $\tilde{\psi}$ signify the components of an elastic curvature caused by the moments $\tilde{M}_{x1}$ and $\tilde{M}_{y1}$; they are obtained from equalities (1.1)

after replacing M_{x1} and M_{y1} by $\tilde{M}_{x1}$ and $\tilde{M}_{y1}$. The calculation by equation (1.7) is simple, since it contains only two integral operators (with an accuracy up to a factor). In a number of practical problems (for example, in calculating the blades of turbomachines) it is possible to ignore the flexure in the plane of maximum rigidity, since $J_\xi > J_\eta$.

In this case, from equations (1.1) we obtain

$$\frac{d^2u}{dr^2} = \frac{M_\eta}{EJ_\eta}\cos\alpha; \qquad (1.10)$$

$$\frac{d^2v}{dr^2} = \frac{M_\eta}{EJ_\eta}\sin\alpha,$$

where

$$M_\eta = M_{y1}\cos\alpha - M_{x1}\sin\alpha \qquad (1.11)$$

is the bending moment relative to axis with a minimal moment of inertia.

From equalities (1.11) and (1.5) there ensues

$$M_\eta = \tilde{M}_\eta - \omega^2 \sin\alpha(r)\int_r^R C_1(r_1)\frac{dv}{dr_1}(r_1)\,dr_1 -$$

$$-\omega^2\cos\alpha(r)\,r\int_r^R C_1(r_1)\frac{d}{dr_1}\left(\frac{u(r_1)}{r_1}\right)dr_1. \qquad (1.12)$$

In considering now relationships (1.6) and (1.10), we shall obtain a boundary integral equation relative to the bending moment M_η:

$$M_\eta = \lambda K M_\eta + \tilde{M}_\eta, \qquad (1.13)$$

where

$$\lambda = -\omega^2,$$

$$KM_\eta = \sin\alpha(r)\int_r^R C_1(r_1)\int_{r_0}^{r_1}\frac{M_\eta(r_2)}{EJ_\eta(r_2)}\sin\alpha(r_2)\,dr_2\,dr_1 +$$

$$+\cos\alpha(r)\,r\int_r^R\frac{C_1(r_1)}{r_1^2}\int_{r_0}^{r_1} r_2\frac{M_\eta(r_2)}{EJ_\eta(r_2)}\cos\alpha(r_2)\,dr_2\,dr_1. \qquad (1.14)$$

For a blade, secured to cylindrical hinge, axis of which coincides with axis r_1 for the root section, the integral equation will be such:

$$M_\eta = \lambda K^* M_\eta + \tilde{M}^*_\eta. \qquad (1.15)$$

where $\lambda = -\omega^2$;

$$K^* M_\eta = KM_\eta - \frac{B(r)}{B(r_0)} KM_\eta \Big|_{r=r_0}; \qquad (1.16)$$

$$\widetilde{M}_\eta^* = \widetilde{M}_\eta(r) - \frac{B(r)}{B(r_0)} \widetilde{M}_\eta(r_0),$$

where KM_η and M_η have the former meaning, and

$$B(r) = \tan\alpha(r_0) \sin\alpha(r) \int_r^R C_1(r_1)\, dr_1 + \cos\alpha(r)\, r r_0 \int_r^R \frac{C_1(r_1)}{r_1^2} dr_1.$$

A further simplification can be attained, if it is assumed

$$\tan\alpha = ar \qquad (1.17)$$

(vane or blade of constant screw pitch*).

Coefficient a may be selected equal to:

$$a = \frac{\tan\alpha_{cp}}{r_{cp}},$$

where r_{cp} is the average radius; $\alpha_{cp} = \alpha(r_{cp})$.

The calculations showed that replacement of real angle of installation of profile in vane of a turbomachine by an angle, determinable from equality (1.17), does not introduce a noticeable error.

We shall have

$$\frac{d^2v}{dr^2} = \frac{d^2u}{dr^2} ar.$$

By integrating both sides of equality from r_0 to r, we obtain for a rigidly fixed blade

$$\frac{dv}{dr} = a\left(r\frac{du}{dr} - u\right) = ar^2 \frac{d}{dr}\left(\frac{u}{r}\right).$$

The latter expression is valid also for a blade fastened on hinges by virtue of the equality

$$\frac{dv}{dr}(r_0) = \frac{du}{dr}(r_0) \tan\alpha(r_0).$$

Now from relationships (1.12) and (1.17) it follows

$$M_\eta(r) = \widetilde{M}_\eta(r) - \omega^2 \sin\alpha(r) \int_r^R \left(1 + \frac{1}{a^2 r^2}\right) C_1(r_1) \frac{dv}{dr}(r_1)\, dr_1 =$$

$$= \widetilde{M}_\eta(r) - \omega^2 \sin\alpha(r) \int_r^R \frac{C_1(r_1)}{\sin^2\alpha(r_1)} \frac{dv}{dr}(r_1)\, dr_1. \qquad (1.18)$$

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*D. Yu. Panov, Calculation of a Propeller for Strength, Transactions of Central Aero-Hydrodynamic Institute, No. 288, 1937.

In view of the dependence

$$\frac{dv}{dr}=\int_{r_0}^{r}\varphi(r_1)\,dr_1+\frac{dv}{dr}(r_0)=\int_{r_0}^{r}\frac{M_\eta(r_1)}{EJ_\eta(r_1)}\sin\alpha(r_1)\,dr_1+\frac{dv}{dr}(r_0) \qquad (1.19)$$

relationship (1.18) can be presented in the form of a boundary integral equation. By introducing equality (1.19) for rigidly fixed blade $\left(\frac{dv}{dr}(r_0)=0\right)$ in relationship (1.18), we obtain

$$M_\eta(r)=\tilde{M}_\eta(r)-\omega^2\sin\alpha(r)\int_{r}^{R}\frac{C_1(r_1)}{\sin^2\alpha(r_1)}\int_{r_0}^{r_1}\frac{M_\eta(r_2)}{EJ_\eta(r_2)}\sin\alpha(r_2)\,dr_2\,dr_1. \qquad (1.20)$$

Integral equation for the curvature $\psi(r)$ is obtained from relationship (1.18), if both sides of the equality are divided by $EJ_\eta\frac{1}{\sin\alpha}$ and dependence (1.19) is taken into account:

$$\psi(r)=\tilde{\psi}(r)-\omega^2\frac{\sin^2\alpha(r)}{EJ_\eta(r)}\int_{r}^{R}\frac{C_1(r_1)}{\sin^2\alpha(r_1)}\int_{r_0}^{r_1}\psi(r_2)\,dr_2\,dr_1. \qquad (1.21)$$

Integral equation relative to the angle of rotation has the form

$$\frac{dv}{dr}(r)=\frac{\widetilde{dv}}{dr}(r)-\omega^2\int_{r_0}^{r}\frac{\sin^2\alpha(r_1)}{EJ_\eta}\int_{r_1}^{R}\frac{C_1(r_2)}{\sin^2\alpha(r_2)}\frac{dv}{dr}(r_2)\,dr_2\,dr_1. \qquad (1.22)$$

Equations (1.20-1.22) correspond to one and the same problem, however, they have their own peculiarities from the point of view of practical use. The difference is found to be also the norms of operators, entering into these equations.

Influence of the centrifugal forces on the flexure depends on the dimensionless parameter of the flexibility of rod

$$\nu=\frac{\omega^2}{\omega_1^2},$$

where $\omega^2_1=\lambda_1$ --first eigenvalue of the homogeneous equation (ω_1 is the value of the angular velocity of rotation, with which centrifugal forces, redirected to compression, cause a loss of stability of the vane).

With small values of the parameter of flexibility $(\nu<0,1)$ the effect of elastic deformations of the rod can be ignored by assuming

$$M_\eta=\ddot{M}_\eta.$$

If the bending moment M_η from a lateral load (and initial displacements of axis

of rod)

$$\tilde{M}_1 = cM_{1\,\text{st}},$$

where $M_{1\,\text{st}}$ is the distribution of the bending moments with the first form of the rod's loss of stability; c is a coefficient, then,

$$M_1 = \frac{\tilde{M}_1}{1+\nu},$$

i.e., with large flexibility parameters, the effect of elastic deformations in a field of centrifugal forces may be very great.

The method of solving the boundary integral equations was discussed earlier.

We present results of the calculation, relating to the problem on the flexure of a rod of uniform section under the effect of distributed transverse and axial loads of constant intensity (Table 6).

Table 6. $\frac{M_1(r_0)}{\tilde{M}_1(r_0)}$ Values for a Rod with a Flexibility Parameter $\nu = 2.58$

	(a) Метод решения	(b) Формула	(c) Приближение 1	2	3	(d) Точное решение
(e)	Сложная итерация по Виарда	(5.38)	0,436	0,300	0,426	0,412
(f)	Сложная итерация с переменным пара̀метром	(5.43)	0,280	0,399	0,409	0,412
(g)	Подобная итерация по равенству площадей	(5.52)	0,453	0,407	0,415	0,412
(h)	Подобная итерация по квадратичному отклонению	(5.53)	0,432	0,410	0,413	0,412
(i)	Подобная итерация по равенству функций	(5.55)	0,331	0,362	0,394	0,412

KEY: (a) Method of solution; (b) Formula; (c) Approximation; (d) Accurate solution; (e) Complex iteration by Viarda; (f) Complex iteration with a variable parameter; (g) Similar iteration by equality of areas; (h) Similar iteration by quadratic deviation; (i) Similar iteration by equality of functions.

Note: Numbers of Formulas Indicated are for Chapter 3.

2. Vibrations of Rods

Problem has had numerous engineering applications, especially in turbomachines in calculating the vibration of vanes. Following presentation refers mainly to problems on vibration of vanes, blades of propellers et cetera.

We shall consider at first the natural vibration. The boundary integral equation of flexure vibrations of a cantilever rod (vanes) relative to amplitude

flexures has the form [See Chapter 3, equation (2.22)]

$$\xi(r)=p^2\int_{r_0}^{r}\int_{r_0}^{r_1}\frac{1}{EJ_\eta(r_2)}\int_{r_0}^{R}\int_{r_1}^{R}\rho F(r_4)\,\xi(r_4)\,dr_4\,dr_3\,dr_2\,dr_1, \quad (2.1)$$

where p is the angular frequency of the vibrations.

In deriving this equation, there is used the equality

$$EJ_\eta(r)\frac{d^2\xi(r)}{dr^2}=M_\eta(r) \quad (2.2)$$

$$M_\eta(r)=p^2\int_{r}^{R}\int_{r_1}^{R}\rho F(r_2)\,\xi(r_2)\,dr_2\,dr_1. \quad (2.3)$$

Equation (2.1) is valid at $\alpha(r)$ = const or in a general case in ignoring the influence of natural torsion which is admissible in determining the first frequency.

From relationship (2.2)

$$\xi(r)=\int_{r_0}^{r}\int_{r_0}^{r_1}\frac{M_\eta(r_2)}{EJ_\eta(r_2)}\,dr_2\,dr_1+\frac{d\xi}{dr}(r_0)(r-r_0), \quad (2.4)$$

for a rigid fixing of the root section

$$\frac{d\xi}{dr}(r_0)=0.$$

From equality (2.3) we obtain a boundary integral equation for amplitude bending moment

$$M_\eta(r)=p^2\int_{r}^{R}\int_{r_1}^{R}\rho F(r_2)\int_{r_0}^{r_2}\int_{r_0}^{r_3}\frac{M_\eta(r_4)}{EJ_\eta(r_4)}\,dr_4\,dr_3\,dr_2\,dr_1, \quad (2.5)$$

or in a bridged form

$$M_\eta=p^2KM_\eta. \quad (2.6)$$

With a hinged fastening of the rod from equalities (2.4) and (2.3) we obtain

$$M_\eta=p^2K^*M_\eta,$$

where

$$K^*M_\eta=KM_\eta-\frac{B(r)}{B(r_0)}KM_\eta\Big|_{r=r_0},$$

$$B(r)=\int_{r}^{R}\int_{r_1}^{R}\rho F(r_2)(r_2-r_0)\,dr_2\,dr_1.$$

Methods of solving homogeneous boundary integral equations are reviewed in Chapter 3.

We shall present results of the calculation pertaining to rod of constant section. The frequency is determined by the formula

$$p = k\frac{1}{l^2}\sqrt{\frac{EJ}{\rho F}}. \tag{2.7}$$

The k values are given in Table 7. The values $k_1=3{,}52$; $k_2=22{,}03$; $k_3=61{,}7$, are accurate.

Table 7. Calculation of First and Second Frequencies by Means of Boundary Integral Equations

(a) Метод расчета	(b) Первая частота			(c) Вторая частота			
	(g) Приближение		(h) точное решение	(g) Приближение			(h) точное решение
	1	2		1	2	3	
(d) Сравнение ординат	3,31	3,52	3,52	24,85	22,82	22,90	22,03
(e) Равенство скалярной нормы	3,55	3,52	3,52	23,07	22,76	22,73	22,03
(f) Минимум квадратичного отклонения	3,52	3,52	3,52	22,64	22,91	22,92	22,03

KEY: (a) Method of calculation; (b) First frequency; (c) Second frequency; (d) Comparison of ordinates; (e) Equality of scalar norm; (f) Minimum of Square deviation; (g) Approximation; (h) Accurate solution.

The initial approximation was selected in the form

$$M_{\eta(0)} = 1 - \zeta,$$

where $\zeta = \frac{r - r_0}{l}$, where, l is length of rod.

In the calculation the rod was subdivided into 10 sectors and the integration was made by the trapezoidal rule. Sufficient accuracy is obtained also in dividing the rod into 5 sectors.

Condition of orthogonality for equation (2.1) has the form

$$\int_{r_0}^{l} \xi_i(r)\,\xi_j(r)\,\rho F(r)\,dr = 0 \qquad (i \neq j,\ i, j = 1, 2, 3\ \ldots); \tag{2.8}$$

for equation (2.5) correspondingly

$$\int_{r_0}^{R} \frac{M_{\eta,i} M_{\eta,j}}{EJ_\eta(r)}\,dr = 0. \tag{2.9}$$

The determination of the second frequency and form of vibrations, is pointed out in Sec. 3, Chap. 3. Thus, for example, equation relative to bending moments

has the form

$$M_\eta = p^2 K_1 M_\eta,$$

$$K_1 M_\eta = K M_\eta - M_{\eta,1} \frac{\int\limits_{r_0}^{R} K M_\eta M_{\eta,1} \frac{1}{EJ_\eta} dr}{\int\limits_{r_0}^{R} M_{\eta,1}^2 \frac{1}{EJ_\eta} dr}. \tag{2.10}$$

Results of the calculation are given in Table 7.

From the table it is evident that the error due to use of trapezoidal rule (with 10 sectors) is larger than error from an "incomplete" convergence of the process.

Application of normal integral equations

$$\xi = p^2 N\xi + \xi(R) f_0 + \frac{d\xi}{dr}(R) f_1 \tag{2.11}$$

has been considered in Sec. 4, Chapter 3.

For a rod of constant section we obtain

$$\xi(\zeta) = p^2 \frac{\rho F l^4}{EJ_\eta} \int\limits_{\zeta}^{1}\int\limits_{\zeta_1}^{1}\int\limits_{\zeta_2}^{1}\int\limits_{\zeta_3}^{1} \xi(\zeta_4)\, d\zeta_4\, d\zeta_3\, d\zeta_2\, d\zeta_1 +$$
$$+ \xi(1)\, 1 - \frac{d\xi}{dr}(1)\, l(1-\zeta); \quad \left(\zeta = \frac{r - r_0}{l}\right).$$

The functions $\Phi_0(\zeta)$ and $\Phi_1(\zeta)$ are equal

$$\Phi_0(\zeta) = 1 + \chi \frac{(1-\zeta)^4}{4!} + \chi^2 \frac{(1-\zeta)^8}{8!} + \dots,$$
$$\Phi_1(\zeta) = -l\left(1 - \zeta + \chi \frac{(1-\zeta)^5}{5!} + \chi^2 \frac{(1-\zeta)^9}{9!} + \dots\right). \tag{2.12}$$

where

$$\chi = p^2 \frac{l^4 \rho F}{EJ}.$$

The frequency equation for the first approximation [in the series (2.12) are retained by two of the first terms]:

$$\begin{vmatrix} 1+\frac{\chi}{4!}, & -l\left(1+\frac{\chi}{5!}\right) \\ -\frac{1}{l}\frac{\chi}{3!}, & 1+\frac{\chi}{4!} \end{vmatrix} = 1-0{,}08334\chi+0{,}000347\chi^2=0.$$

The first approximation for the first frequency

$$p_{(1)}=\sqrt{\frac{1}{0{,}08334}}\frac{1}{l^2}\sqrt{\frac{EJ}{\rho F}}=3{,}464\frac{1}{l^2}\sqrt{\frac{EJ}{\rho F}}$$

differs from the accurate by 1.7%.

In the second approximation for first three frequencies, we obtain values of the coefficient k [equality (2.7)]:

$$k_{1(2)}=3{,}52;\quad k_{2(2)}=21{,}3;\quad k_{3(2)}=33{,}6.$$

In the third approximation

$$k_{1(3)}=3{,}52;\quad k_{2(3)}=21{,}95;\quad k_{3(3)}=65{,}13.$$

We shall consider now the vibrations of a naturally twisted rod.

If $u(r)$ and $v(r)$ are the amplitude displacements of points on the vane's axis along the axes x and y, then the bending moments

$$M_{x1}(r)=-p^2\int_r^R\int_{r_1}^R \rho F(r_2)\,v(r_2)\,dr_2\,dr_1$$

and

$$M_{y1}(r)=p^2\int_r^R\int_{r_1}^R \rho F(r_2)\,u(r_2)\,dr_2\,dr_1.$$

In accordance with equality (1.11) we obtain

$$M_\eta(r)=p^2\Big(\cos\alpha\int_r^R\int_{r_1}^R \rho F(r_2)\,u(r_2)\,dr_2\,dr_1+ \qquad (2.13)$$
$$+\sin\alpha\int_r^R\int_{r_1}^R \rho F(r_2)\,v(r_2)\,dr_2\,dr_1\Big).$$

By integrating equations (1.10) and (1.11) and introducing the result into equality (2.13), we shall have

$$M_\eta(r)=p^2\Big(\cos\alpha\int_r^R\int_{r_1}^R \rho F(r_2)\int_{r_0}^{r_2}\int_{r_0}^{r_3}\frac{M_\eta(r_4)}{EJ_\eta(r_4)}\cos\alpha(r_4)\,dr_4\,dr_3\,dr_2\,dr_1+ \qquad (2.14)$$
$$+\sin\alpha\int_r^R\int_{r_1}^R \rho F(r_2)\int_{r_0}^{r_2}\int_{r_0}^{r_3}\frac{M_\eta(r_4)}{EJ_\eta(r_4)}\sin\alpha(r_4)\,dr_4\,dr_3\,dr_2\,dr_1\Big).$$

If angle $\alpha(r)$ = const, then equations (2.14) and (2.[illegible]) coincide.

At an angle of natural torsion of rod of an order of 30–40°, the torsion insignificantly increases the first frequency (to 1%) and considerably lowers the second (15 to 20%); this is confirmed by experimental data.

The determination of the second frequency is made by equation (2.10) with the operator KM_{η}, corresponding to equation (2.14).

In calculating both rigidities for flexure the problem reduces to a homogeneous, boundary, matrix integral equation.

We shall present for this case the conditions of orthogonality

$$\int_{r_0}^{R}(u_iu_j+v_iv_j)\rho F\,dr=0,$$

$$\int_{r_0}^{R}\left(\frac{M_{\eta,i}M_{\eta,j}}{EJ_{\eta}}+\frac{M_{\xi,i}M_{\xi,j}}{EJ_{\xi}}\right)dr=0$$

$$(i\neq j,\quad i,j=1,2,3,\ \ldots)$$

The calculation of torsional vibrations of rods also reduces to solution of homogeneous boundary or normal integral equations.

The boundary integral equation for amplitude angles of rotation has the form

$$\theta(r)=p^2\int_{r_0}^{r}\frac{1}{GT(r_1)}\int_{r_1}^{R}\rho J_{pl}(r_2)\,\theta(r_2)\,dr_2\,dr_1, \qquad (2.15)$$

where $GT(r)$ is strength of section of rod to torsion; $J_{pl}(r)$ is the polar moment of inertia of section relative to the center of rigidity.

Corresponding equation for amplitude torques is such.

$$M_k(r)=p^2\int_{r}^{R}\rho J_{pl}(r_1)\int_{r_0}^{r_1}\frac{M_k(r_2)}{GT(r_2)}\,dr_2\,dr_1. \qquad (2.16)$$

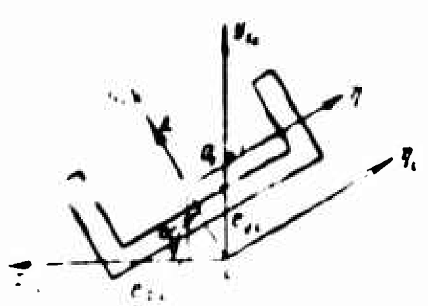

Fig. 11. For deriving equations of flexure-torsional vibrations of rod.

Conditions of orthogonality are written out in the form

$$\int_{r_0}^{R} \theta_i \theta_j \rho J_{pi}\, dr = 0; \qquad \int_{r_0}^{R} \frac{M_{\kappa i} M_{\kappa j}}{GT}\, dr = 0.$$

The normal equation for torsional vibrations

$$\theta(r) = -p^2 \int_{r}^{R} \frac{1}{GT(r_1)} \int_{r}^{R} \rho J_{pi}(r_2)\, \theta(r_2)\, dr_2\, dr_1 + \theta(R).$$

We turn now to the calculation of flexure-torsional vibrations of a rod (vane).

The origin of the local system of coordinates is placed at the center of rigidity of section; coordinates of center of gravity in this coordinate system will be e_{xi}, e_{yi} (Fig. 11).

In designating the amplitude displacement of center of rigidity u_i and v_i, angle of rotation θ, we obtain a system of three differential equations

$$\frac{d^2}{dr^2}\left\{-(EJ_\xi - EJ_\eta)\sin\alpha\cos\alpha\,\frac{d^2u_i}{dr^2} + (EJ_\xi\cos^2\alpha + EJ_\eta\sin^2\alpha)\,\frac{d^2v_i}{dr^2}\right\} =$$

$$= p^2(v_i\rho F + \theta_i\rho F e_{xi});$$

$$\frac{d^2}{dr^2}\left\{(EJ_\xi\sin^2\alpha + EJ_\eta\cos^2\alpha)\,\frac{d^2u_i}{dr^2} - (EJ_\xi - EJ_\eta)\sin\alpha\cos\alpha\,\frac{d^2v_i}{dr^2}\right\} =$$

$$= p^2(u_i\rho F - \theta_i\rho F e_{yi}); \tag{2.17}$$

$$\frac{d}{dr}\left(GT\frac{d\theta}{dr}\right) = -p^2(\theta\rho J_{pi} + v_i\rho F e_{xi} - u_i\rho F e_{yi}).$$

This system is equivalent to the homogeneous matrix boundary integral equation

$$\begin{bmatrix} \varphi \\ \psi \\ \theta \end{bmatrix} = p^2 \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \\ \theta \end{bmatrix}. \tag{2.18}$$

where

$$\varphi = \frac{d^2u_i}{dr^2}, \quad \psi = \frac{d^2v_i}{dr^2}.$$

Equation (2.18) is obtained by previously indicated methods, and the value of operators is not written out here.

In calculating vanes it is possible usually to ignore the flexure in plane of the maximum rigidity, whereas the flexure-torsional vibrations are described by the

following equation relative to the bending and turning moments:

$$\begin{bmatrix} M_\eta \\ M_\kappa \end{bmatrix} = p^2 \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} M_\eta \\ M_\kappa \end{bmatrix}, \tag{2.19}$$

where

$$K_{11}M_\eta = \cos\alpha \int_r^R \int_{r_1}^R \rho F(r_2) \int_{r_0}^{r_2} \int_{r_0}^{r_3} \frac{M_\eta(r_4)}{EJ_\eta(r_4)} \cos\alpha(r_4)\, dr_4\, dr_3\, dr_2\, dr_1 +$$

$$+ \sin\alpha \int_r^R \int_{r_1}^R \rho F(r_2) \int_{r_0}^{r_2} \int_{r_0}^{r_3} \frac{M_\eta(r_4)}{EJ_\eta(r_4)} \sin\alpha(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$K_{12}M_\kappa = \sin\alpha \int_r^R \int_{r_1}^R e_{xi}(r_2)\, \rho F(r_2) \int_{r_0}^{r_2} \frac{M_\kappa(r_3)}{GT(r_3)}\, dr_3\, dr_2\, dr_1 -$$

$$- \cos\alpha \int_r^R \int_{r_1}^R e_{yi}(r_2)\, \rho F(r_2) \int_{r_0}^{r_2} \frac{M_\kappa(r_3)}{GT(r_3)}\, dr_3 dr_2 dr_1;$$

$$K_{21}M_\eta = \int_r^R e_{xi}(r_1) \rho F(r_1) \int_{r_0}^{r_1} \int_{r_0}^{r_2} \frac{M_\eta(r_3)}{EJ_\eta(r_3)} \sin\alpha(r_3)\, dr_3\, dr_2\, dr_1 -$$

$$- \int_r^R e_{yi}(r_1) \rho F(r_1) \int_{r_0}^{r_1} \int_{r_0}^{r_2} \frac{M_\eta(r_3)}{EJ_\eta(r_3)} \cos\alpha(r_3)\, dr_3\, dr_2\, dr_1;$$

$$K_{22}M_\kappa = \int_r^R \rho J_{pi}(r_1) \int_{r_0}^{r_1} \frac{M_\kappa(r_2)}{GT(r_2)}\, dr_2\, dr_1.$$

The calculation is made by the equation (See Chap. 3, Sec. 3)

$$\begin{bmatrix} M_{\eta(i)} \\ M_{\kappa(i)} \end{bmatrix} = p^2_{(i)} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} M_{\eta(i-1)} \\ M_{\kappa(i-1)} \end{bmatrix}.$$

Under the condition of normalization by a scalar norm

$$p^2_{(i)} = \frac{\sqrt{\int_{r_0}^R \left(\frac{M^2_{\eta(i-1)}}{EJ_\eta} + \frac{M^2_{\kappa(i-1)}}{GT} \right) dr}}{\sqrt{\int_{r_0}^R \left[\frac{(K_{11}M_{\eta(i-1)} + K_{12}M_{\kappa(i-1)})^2}{EJ_\eta} + \frac{(K_{21}M_{\eta(i-1)} + K_{22}M_{\kappa(i-1)})^2}{GT} \right] dr}}.$$

Second frequency of flexure-torsional vibrations is determined from the equation

$$\begin{bmatrix} M_\eta \\ M_\kappa \end{bmatrix} = p^2 \left\{ \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} M_\eta \\ M_\kappa \end{bmatrix} - \beta_1 \begin{bmatrix} M_{\eta,1} \\ M_{\kappa,1} \end{bmatrix} \right\},$$

where

$$\beta_1 = \frac{\int\limits_{r_0}^{R} \left[(K_{11}M_\eta + K_{12}M_\kappa) \frac{M_{\eta,1}}{EJ_\eta} + (K_{21}M_\eta + K_{22}M_\kappa) \frac{M_{\kappa,1}}{GT} \right] dr}{\int\limits_{r_0}^{R} \left(\frac{M_{\eta,1}^2}{EJ_\eta} + \frac{M_{\kappa,1}^2}{GT} \right) dr}.$$

We now consider the flexure vibrations of a naturally twisted rod in a field centrifugal forces. They are described by the following system of differential equations with respect to amplitude flexures:

$$\frac{d^2}{dr^2}\left[-(EJ_\xi - EJ_\eta)\sin\varepsilon\cos\varepsilon \frac{d^2u}{dr^2} + (EJ_\xi \cos^2\varepsilon + EJ_\eta \sin^2\varepsilon) \frac{d^2v}{dr^2} \right] =$$

$$= p^2\rho F v + \frac{d}{dr}\left(C(r) \frac{dv}{dr} \right);$$

$$\frac{d^2}{dr^2}\left[(EJ_\xi \sin^2\varepsilon + EJ_\eta \cos^2\varepsilon) \frac{d^2u}{dr^2} - (EJ_\xi - EJ_\eta)\sin\varepsilon\cos\varepsilon \frac{d^2v}{dr^2} \right] = \qquad (2.20)$$

$$= p^2\rho F u + \frac{d}{dr}\left(C(r) \frac{du}{dr} \right) + \rho\omega^2 F u,$$

where $C(r)$ is the tension in the section of the rod.

In the particular case, we will have vibrations of an untwisted vane in axial plane

$$\frac{d^2}{dr^2}\left(EJ_\eta \frac{d^2v}{dr^2} \right) = p^2\rho F v + \frac{d}{dr}\left(C(r) \frac{dv}{dr} \right).$$

by integrating this equality, we find

$$EJ_\eta \frac{d^2v}{dr^2}(r) = p^2 \int\limits_r^R \int\limits_{r_1}^R \rho F(r_2)\, v(r_2)\, dr_2\, dr_1 - \qquad (2.21)$$

$$- \omega^2 \int\limits_r^R C_1(r_1) \frac{dv}{dr}(r_1)\, dr_1.$$

In considering relationship of form (1.4), we shall obtain a boundary integral equation for a rigidly secured rod

$$M_\eta = p^2 K_\rho M_\eta - \omega^2 K_\omega M_\eta. \qquad (2.22)$$

where

$$K_pM_\eta = \int_r^R \int_{r_1}^R \rho F(r_2) \int_{r_0}^{r_2} \int_{r_0}^{r_3} \frac{M_\eta(r_4)}{EJ_\eta(r_4)} dr_4\, dr_3\, dr_2\, dr_1;$$

$$K_\omega M_\eta = \int_r^R C_1(r_1) \int_{r_0}^{r_1} \frac{M_\eta(r_2)}{EJ_\eta(r_2)} dr_2\, dr_1.$$

Equation (2.22) is a two-parametric integral equation.

It is possible to show that if equation of flexure of vane

$$M_\eta = \tilde{M}_\eta - \omega^2 K_\omega M_\eta,$$

where $\tilde{M}_\eta$ is the bending moment of the transverse load, and the equation

$$M_\eta = p^2 K_p M_\eta$$

corresponds to the problem on vibrations of non-rotating rod, then integral equation of vibration of rod in a field of centrifugal forces has the form (2.22).

In composing equation (2.22) there may be made assumptions of a different physical nature relative to the operators $K M_\eta$ and $K_\omega M_\eta$. For example, in determining the first frequency, operator K_pM_η can be taken without considering the natural torsion (2.5), and the operator $K_\omega M_\eta$ may be taken the same, as for a vane of constant screw pitch (1.20).

In solving equation (2.22) by the method of successive approximations we obtain

$$M_{\eta(i)} = p^2_{(i)} K_p M_{\eta(i-1)} - \omega^2 K_\omega M_{\eta(i-1)}. \qquad (2.23)$$

By using the norm of function on basis of maximum (method of comparing ordinates) we find

$$p^2_{(i)} = \left. \frac{M_{\eta(i-1)} + \omega^2 K_\omega M_{\eta(i-1)}}{K_p M_{\eta(i-1)}} \right|_{r=r_0}.$$

The process of successive approximations is unconditionally convergent with the parameter of flexibility $\nu < 1$. For more flexible vanes one should apply corresponding iterative processes. Thus, by applying methods, similar iterations, we obtain an equation of ordinary structure

$$M_{\eta(i)} = p^2_{(i)} K M_{\eta(i-1)}, \qquad (2.24)$$

where by similar iteration on basis of equality of functions

$$KM_{\eta(i-1)}=\frac{K_pM_{\eta(i-1)}}{M_{\eta(i-1)}+\omega^2K_\omega M_{\eta(i-1)}}M_{\eta(i-1)},$$

by similar iteration on basis of equality of areas

$$KM_{\eta(i-1)}=K_pM_{\eta(i-1)}-\omega^2\frac{\int\limits_{r_0}^{R}K_pM_{\eta(i-1)}dr}{\int\limits_{r_0}^{R}(M_{\eta(i-1)}+\omega^2K_\omega M_{\eta(i-1)})\,dr}K_\omega M_{\eta(i-1)},$$

by similar iteration on basis of minimum of square deviation

$$KM_{\eta(i-1)}=K_pM_{\eta(i-1)}-\omega^2\frac{\int\limits_{r_0}^{R}K_pM_{\eta(i-1)}(M_{\eta(i-1)}+\omega^2K_\omega M_{\eta(i-1)})\,dr}{\int\limits_{r_0}^{R}(M_{\eta(i-1)}+\omega^2K_\omega M_{\eta(i-1)})^2dr}\times$$
$$\times K_\omega M_{\eta(i-1)}.$$

In solving equation (2.24) there are used methods, indicated for the vibration of non-rotating rods.

Above were considered the natural vibrations of rods. Let us turn now to problem about forced vibrations, at first without taking into account the forces of damping.

As example we shall take the flexure vibrations of a rod.

Suppose onto the rod is applied an external excitation load.

$$q=q(r)\cos\nu t.$$

The equation for amplitude flexures of forced vibrations has the form

$$\xi(r)=\nu^2\int\limits_{r_0}^{r}\int\limits_{r_0}^{r_1}\frac{1}{EJ_\eta(r_2)}\int\limits_{r_2}^{R}\int\limits_{r_3}^{R}\rho F(r_4)\xi(r_4)\,dr_4\,dr_3\,dr_2\,dr_1+$$
$$+\int\limits_{r_0}^{r}\int\limits_{r_0}^{r_1}\frac{1}{EJ_\eta(r_2)}\int\limits_{r_2}^{R}\int\limits_{r_3}^{R}q(r_4)\,dr_4\,dr_3\,dr_2\,dr_1$$

or in abridged form

$$\xi=\nu^2K\xi+f. \qquad (2.25)$$

where f is the flexure of vane's axis under action of distributed load q(r).

Process of simple iteration

$$\xi_{(i)} = \nu^2 K \xi_{(i-1)} + f$$

converges, if $\nu^2 < p_1^2$, where p_1 is the first natural frequency.

At $\nu > p_1$ one should apply the previously indicated iterative processes.

Rapidity of convergence depends on form function f and at $\nu < 4p_1$ it is obtained usually entirely satisfactory.

The very best results in a number of practical examples were given by the method of similar iteration on basis of minimum of square deviation.

We now consider question of determining the coefficients of dynamic rigidity of rod of variable section.

Suppose onto the root section of rod there are applied dynamic (Fig. 12).

$$Q = Q_0 \cos \nu t,$$
$$M = M_0 \cos \nu t.$$

This will cause a vibration of entire vane, with which in root section there will be a flexure ξ and angle of rotation $\frac{\partial \xi}{\partial r}$:

$$\xi = \xi_0 \cos \nu t$$
$$\frac{\partial \xi}{\partial r} = \xi_0' \cos \nu t.$$

There exist the linear relationships

$$Q_0 = a_{11}\xi_0 + a_{12}\xi_0', \qquad (2.26)$$
$$M_0 = a_{21}\xi_0 + a_{22}\xi_0'.$$

The coefficients a_{ij} are called coefficients of dynamic rigidity. They possess the property of reciprocity

$$a_{ij} = a_{ji}.$$

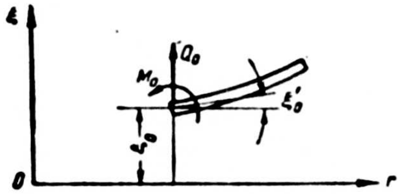

Fig. 12. Determination of Coefficients of Dynamic Rigidity.

We shall write out the normal integral equation for considered case in following form:

$$\xi = \nu^2 N\xi + \xi_0 f_0 + \xi_0' f_1 + M_0 f_2 + Q_0 f_3, \tag{2.27}$$

where

$$N\xi = \int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta(r_2)} \int_{r_0}^{r_2}\int_{r_0}^{r_3} \rho F(r_4)\, \xi(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$f_0 = 1; \quad f_1 = r - r_0; \quad f_2 = -\int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta} dr_2\, dr_1.$$

The solution of equation (2.27) will be such:

$$\xi = \xi_0 \Phi_0 + \xi_0' \Phi_1 + M_0 \Phi_2 + Q_0 \Phi_3, \tag{2.28}$$

where

$$\Phi_i = f_i + \nu^2 N f_i + \nu^4 N^2 f_i + \ldots \quad (i = 0, 1, 2, 3).$$

By introducing the values (2.28) into the boundary conditions

$$Q(R) = 0; \quad M(R) = 0,$$

we arrive at equalities (2.26), where the coefficients a_{ij} become known.

Let us consider now forced vibrations with a consideration of linear damping.

The differential equation of the problem is written out as:

$$\frac{\partial^2}{\partial r^2}\left(EJ_\eta \frac{\partial^2 \xi}{\partial r^2}\right) + h(r)\frac{\partial \xi}{\partial t} + \rho F(r)\frac{\partial^2 \xi}{\partial t^2} = q(r)\cos \nu t.$$

By putting the solution in the form

$$\xi = w(r)\cos \nu t + z(r)\sin \nu t,$$

We arrive at a system of equations

$$\begin{aligned} &\frac{d^2}{dr^2}\left(EJ_\eta \frac{d^2 w(r)}{dr^2}\right) - \nu^2 \rho F(r)\, w(r) + \nu h(r)\, z(r) = q(r), \\ &\frac{d^2}{dr^2}\left(EJ_\eta \frac{d^2 z(r)}{dr^2}\right) - \nu h(r)\, w(r) - \nu^2 \rho F(r)\, z(r) = 0. \end{aligned} \tag{2.29}$$

For a rigidly secured vane this system is equivalent to a inhomogeneous boundary matrix equation

$$\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{2.30}$$

where

$$K_{11}w = \nu^2 \int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta(r_2)} \int_{r_2}^{R}\int_{r_3}^{R} \rho F(r_4)\, w(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$K_{12}z = -\nu \int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta(r_2)} \int_{r_2}^{R}\int_{r_3}^{R} h(r_4)\, z(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$K_{21}w = \nu \int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta(r_2)} \int_{r_2}^{R}\int_{r_3}^{R} h(r_4)\, w(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$K_{22}z = \nu^2 \int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta(r_2)} \int_{r_2}^{R}\int_{r_3}^{R} \rho F(r_4)\, z(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$f_1 = \int_{r_0}^{r}\int_{r_0}^{r_1} \frac{1}{EJ_\eta(r_2)} \int_{r_2}^{R}\int_{r_3}^{R} q(r_4)\, dr_4\, dr_3\, dr_2\, dr_1;$$

$$f_2 = 0.$$

Equation (2.30) at $\nu < p_1$ is solved by the method of simple iteration, at $\nu > p_1$ there can be applied the method of similar iteration (See Sec. 4, Chap 3).

3. Critical Speed of Shafts

The determination of critical speed is important for many high speed machines, expecially turbomachines.

Let us consider the general case of the precessional motion of shaft (Fig. 13). Suppose the plane, containing elastic line of shaft, revolves with angular velocity ν, and the shaft itself is rotated in reference to this plane with an angular velocity

The kinematic model of a similar motion is shown in Fig. 14.

Angular velocity of shaft is equal to

$$\omega = \nu + \lambda. \tag{3.1}$$

The angular velocities ν and λ are presented in the following form:

$$\nu = \varepsilon\omega,$$
$$\lambda = (1-\varepsilon)\omega, \tag{3.2}$$

where ε perhaps, in general, is an arbitrary (real) number*.

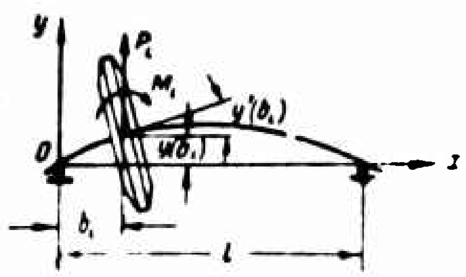

Fig. 13. Precessional motion of shaft.

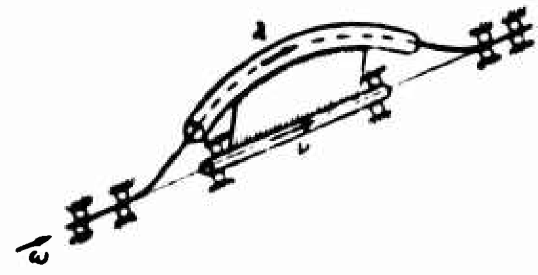

Fig. 14. Kinematic model of precessional motion of shaft.

The disk acts on the shaft with a stress and a moment

$$P_i = \omega^2 \alpha_i y(b_i), \tag{3.3}$$
$$M_i = \omega^2 \beta_i y'(b_i),$$
$$\alpha_i = \varepsilon^2 m_i; \quad \beta_i = \varepsilon(2-\varepsilon) I_i, \tag{3.4}$$

where m_i is the mass of disk; $I_i = \frac{1}{2} I_{pi}$ is the equatorial moment of inertia of disk.

The equation of the stability of revolving shaft with distributed masses $m(x)$ and moments of inertia $I(x)$ has the form

$$\frac{d^2}{dx^2}\left(EJ(x)\frac{d^2y}{dx^2}\right) - \omega^2\left[\varepsilon^2 m(x) y(x) + \varepsilon(2-\varepsilon)\frac{d}{dx}(I(x) y'(x))\right] = 0. \tag{3.5}$$

Equation of flexure vibrations of shaft with a calculation of inertia of turn

$$\frac{d^2}{dx^2}\left(EJ(x)\frac{d^2y}{dx^2}\right) - \omega^2\left[m(x) y(x) - \frac{d}{dx}\left(I(x) y'(x)\right)\right] = 0. \tag{3.6}$$

At $I(x) = 0$ equations (3.5) and (3.6) coincide.

Solution of equation (3.5) can be used and as solution of equation (3.6), if it is assumed $\varepsilon = -1$ (reverse synchronous procession) and to decrease in the magnitude of $I(x)$ by three times.

We now turn to composing integral equations of the problem. Let us consider

---

*The precession is called synchronous, if $|\omega| = |\nu|$. In an identical direction ω and ν the precession is considered forward, with a different direction--reverse.

as an example a shaft on two pivoting bearings (Fig. 13), carrying the distributed masses m(x). The solution can be applied also in the presence of individual masses, if the mass of the disk is distributed along the length of corresponding section. The gyroscopic effect we disregard, and the parameter $\varepsilon=1$. Bending moment in section x will be equal to

$$M(x)=R_1x+\omega^2\int_0^x\int_0^{x_1} m(x_2)\,y(x_2)\,dx_2\,dx_1.$$

By determining reaction in the left support from condition M(l) = 0, we find

$$M(x)=\omega^2 A_y,$$

where

$$A_y=\int_0^x\int_0^{x_1} m(x_2)\,y(x_2)\,dx_2\,dx_1-\frac{x}{l}\int_0^l\int_0^{x_1} m(x_2)\,y(x_2)\,dx_2\,dx_1.$$

By using the equation of flexure $\frac{d^2y}{dx^2}=\frac{M(x)}{EJ(x)}$, we obtain,

$$y(x)=\omega^2\int_0^x\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1+y'(0)\,x+y(0). \qquad (3.7)$$

By determining $y'(0)$ from condition $y(l)=0$ (magnitude $y(0)=0$), we arrive at a homogeneous boundary integral equation

$$y=\omega^2 Ky.$$

$$Ky=\int_0^x\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1-\frac{x}{l}\int_0^l\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1. \qquad (3.8)$$

We now consider the general case (Fig. 15), when there are considered the distributed and concentrated masses. The transverse force in section of x

$$Q(x)=\int_0^x q(x_1)\,dx_1+\sum_{i=1}^{n} S(x,b_i)\,P_i+\sum_{i=1}^{2} S(x,a_i)\,R_i, \qquad (3.9)$$

where the unit function, for example, $S(x, b_i)$, is determined by the equality

$$S(x,b_i)=\begin{cases}0 & x\leqslant b_i,\\ 1 & x>b_i.\end{cases}$$

In considering the dependencies

$$q(x)=\omega^2\alpha(x)\,y(x);\quad \alpha(x)=\varepsilon^2 m(x),$$

$$P_i=\omega^2\alpha_i y(b_i),$$

We shall write out expression (3.8) in the following form:

$$Q(x)=\omega^2 A_{1y}(x)+\sum_{i=1}^{2} S(x,a_i)\,R_i \qquad (3.10)$$

and for the bending moment

$$M(x) = \omega^2 A_{2y}(x) + \sum_{i=1}^{2} S(x, a_i) R_i (x - a_i), \tag{3.11}$$

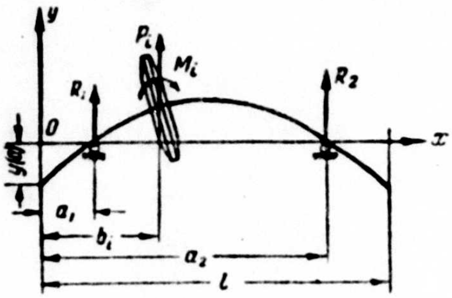

Fig. 15. Shaft on two pivoting bearings.

where

$$A_{1y}(x) = \int_0^x \alpha(x_1) y(x_1) dx_1 + \sum_{i=1}^{n} S(x, b_i) \alpha_i y(b_i),$$

$$A_{2y}(x) = \int_0^x \int_0^{x_1} \alpha(x_2) y(x_2) dx_2 dx_1 + \int_0^x \beta(x_1) y'(x_1) dx_1 + \sum_{i=1}^{n} S(x, b_i) [\alpha_i y(b_i)(x - b_i) + \beta_i y'(b_i)] \tag{3.12}$$

$$[\beta(x) = \varepsilon(2 - \varepsilon) I(x)].$$

In determining reaction R_1 and R_2 from conditions

$$Q(l) = 0, \quad M(l) = 0, \tag{3.13}$$

we obtain

$$M(x) = \omega^2 A_y(x),$$

where

$$A_y(x) = A_{2y}(x) + S(x, a_1) \frac{x - a_1}{a_2 - a_1} [A_{1y}(l)(l - a_2) - A_{2y}(l)] + S(x, a_2) \frac{x - a_2}{a_2 - a_1} [A_{2y}(l) - A_{1y}(l)(l - a_1)]. \tag{3.14}$$

In determining in equality (3.7) the magnitudes $y(0)$ and $y'(0)$ from conditions $y(a_1) = 0$ and $y(a_2) = 0$, we obtain a system of integral equations

$$y = \omega^2 K_0^{(1)} y, \qquad y^{(1)} = K_1^{(1)} y, \tag{3.15}$$

where

$$K_0^{(1)} y = \int_0^x \int_0^{x_1} \frac{A_y(x_2)}{EJ(x_2)} dx_2 dx_1 + \frac{x}{a_2 - a_1} \left[\int_0^{a_1} \int_0^{x_1} \frac{A_y(x_2)}{EJ(x_2)} dx_2 dx_1 - \int_0^{a_2} \int_0^{x_1} \frac{A_y(x_2)}{EJ(x_2)} dx_2 dx_1 \right] + \frac{1}{a_2 - a_1} \left[a_1 \int_0^{a_2} \int_0^{x_1} \frac{A_y(x_2)}{EJ(x_2)} dx_2 dx_1 - \right.$$

$$-a_2\int_0^{a_1}\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1\Bigg];\qquad (3.16)$$

$$K_1^{(1)}y=\int_0^{x}\frac{A_y(x_1)}{EJ(x_1)}\,dx_1+\frac{1}{a_2-a_1}\Bigg[\int_0^{a_1}\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1$$

$$-\int_0^{a_2}\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1\Bigg].\qquad (3.17)$$

Calculation by formulas (3.16) and (3.17) are very simple, since they contain all two integral operations

$$\int_0^{x}\frac{A_y(x_1)}{EJ(x_1)}\,dx_1\quad\text{and}\quad\int_0^{x}\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}\,dx_2\,dx_1.$$

Equation (3.15) expresses the matrix integral equation

$$\begin{bmatrix} y \\ y^{(1)}\end{bmatrix}=\omega^2\begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11}\end{bmatrix}\begin{bmatrix} y \\ y^{(1)}\end{bmatrix},\qquad (3.18)$$

where

$$K_{00}y+K_{01}y^{(1)}=K_0^{(1)}y,$$

$$K_{10}y+K_{11}y^{(1)}=K_1^{(1)}y.$$

In abridged form, equation (3.15) is written out as:

$$[y]=\omega^2\,[K^{(1)}]\,[y].\qquad (3.19)$$

In solving by the method of successive approximations

$$[y_{(i)}]=\omega_{(i)}^2\,[K^{(1)}]\,[y_{(i-1)}].\qquad (3.20)$$

First line of this equality

$$y_{(i)}=\omega_{(i)}^2K_0^{(1)}y_{(i-1)}.$$

By method of comparison of ordinates

$$\omega_{(i)}^2=\frac{y_{(i-1)}}{K_0^{(1)}y_{(i-1)}}\bigg|_{x=x_{mi}},$$

where x_{mi} is the abscissa of section, corresponding to the maximum value $|y_{(i-1)}|$. After determining $\omega^2_{(i)}$ we find $y_{(i)}$ and $y'_{(i)}$ and further $\omega^2_{(i+1)}$.

Practice of calculation showed that there is sufficient not more than two approximations (second approximation is for control).

Thus, there is determined the minimum in absolute value eigenvalue.

We shall dwell on one circumstance associated with the calculation of systems with strong influence of the gyroscopic effect (for example, with disks, located near supports).

It may be found (in practical cases extremely rarely) that $\bar{\omega}^2_1 < 0$.

This means that real angular velocity will be greater than $|\omega_1|$ and must be determined by taking into account corresponding condition of orthogonality (see determination of second critical speed).

If true value is

$$\omega_1^2 > 0,$$

but in the first approximation in view of unsuccessful selection of $y_{(0)}$ there is obtained

$$\omega^2_{1(0)} < 0,$$

then one should continue the process further and it will converge to a real angular velocity ω_1.

In engineering problems, the indicated cases may be encountered as exceptional, and only with a calculation of gyroscopic effect.

For a number of problems encountered in practice of (rotor with large number of disks, calculation of mass proper of rotor et cetera) the calculations of the discussed method are found to be significantly less laborous, than calculation by other methods (for example, requiring the determination of influence coefficients).

We now turn to determining the second critical angular velocity. Condition of orthogonality in considered problem has the form*

$$\int_0^l [y_i y_j \alpha(x) - y_i^{(1)} y_j^{(1)} \beta(x)]\, dx = 0$$
$$(i \neq j,\ i,j = 1,\ 2,\ 3,\ldots).$$

*In the presence of concentrated masses and moments of inertia, integrals are taken in the sense of Stieltjes.

where $\alpha(x)$ and $\beta(x)$ depend on distribution of the masses and moments of the inertia along the length of shaft

$$\alpha(x)=\varepsilon^2 m(x), \quad \beta(x)=\varepsilon(2-\varepsilon)I(x).$$

The calculation is made by the equation

$$[y]=\omega^2[K_2^{(1)}][y], \tag{3.21}$$

where

$$[K_2^{(1)}][y]=[K^{(1)}]-\beta_1[y_1],$$

$$\beta_1=\frac{([K^{(1)}][y],\ [y_1])^{(1)}}{\|[y_1]\|^2}=\frac{\int\limits_0^l (K_0^{(1)}yy_1\alpha-K_1^{(1)}yy_1'\beta)\,dx}{\int\limits_0^l (y_1^2\alpha-y_1'^2\beta)\,dx}.$$

Above there has been considered a shaft (rotor) on two pivoting bearings.

Analogous equations may be compiled also for other cases. Suppose, for example, the supports of the shaft are elastic, then there exist the dependencies*

$$\left.\begin{aligned} y(a_1)&=-\frac{R_1}{K_{\partial 1}};\\ y(a_2)&=-\frac{R_2}{K_{\partial 2}}, \end{aligned}\right\} \tag{3.22}$$

where $K_{\partial 1}, K_{\partial 2}$ is the rigidity coefficient of the support.

If the support is a complex system of masses and elasticity, then the magnitudes $K_{\partial 1}$ and $K_{\partial 2}$ are dynamic rigidity.

Relationships (3.10) and (3.12) remain in force, and from conditions (3.13) the magnitudes R_1 and R_2 are determined by $A_{1\nu}$ and $A_{2\nu}$.

The magnitude $y(0)$ and $y'(0)$ in equality (3.7) are found from condition (3.22) which results in a corresponding integral equation. There are no great difficulties in composing the equations and in other calculating cases (shaft on several pivoting bearings).

Let us consider as an example, a shaft with large number of identical disks on two pivoting supports (Fig. 16). Per unit of length of shaft, there should be a mass m and moment of inertia I.

*In the presence of a connection between the supports, $y(a_1)$ and $y(a_2)$ are expressed through the linear combinations R_1 and R_2.

Equations (3.15) have the form

$$y(x)=\omega^2\left\{\int_0^x\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}dx_2\,dx_1-\frac{x}{l}\int_0^l\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}dx_2\,dx_1\right\};$$

$$y^{(1)}(x)=\omega^2\left\{\int_0^x\frac{A_y(x_1)}{EJ(x_1)}dx_1-\frac{1}{l}\int_0^l\int_0^{x_1}\frac{A_y(x_2)}{EJ(x_2)}dx_2\,dx_1\right\}.$$

By virtue of equality (3.14) and (3.12)

$$A_y(x)=A_{2y}(x)-\frac{x}{l}A_{2y}(l);$$

$$A_{2y}(x)=\int_0^x\int_0^{x_1}\alpha(x_2)\,y(x_2)\,dx_2\,dx_1+\int_0^x\beta(x_1)\,y'(x_1)\,dx_1,$$

where for forward synchronous precession

$$\alpha(x)=m;\quad \beta(x)=I.$$

As an initial approximation $y_{(0)}(x)$ for a shaft on two pivoting bearings, it is possible to take

$$y_{(0)}=C(x-a_1)(x-a_2), \qquad (3.23)$$

where the constant C expediently is determined from the condition $y_{(0)\max}=1$.

In accordance with equality (3.23)

$$y'_{(0)}(x)=C[2x-(a_1+a_2)].$$

Shaft was subdivided into ten sectors and the integration was made by the trapezoidal rule. The ratio is $\frac{d}{l}=\frac{1}{2}$.

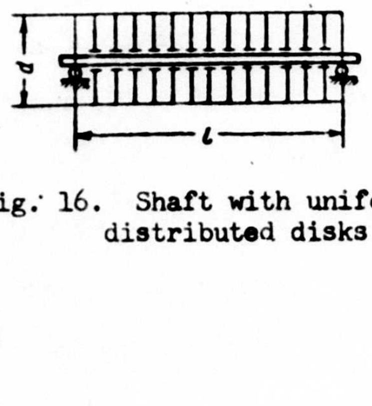

Fig. 16. Shaft with uniformly distributed disks.

In the first approximation there was obtained

$$\omega_{(1)}=10{,}76\sqrt{\frac{EJ}{ml^4}}$$

(if there were made an accurate integration $\omega_{(1)}=10{,}56\sqrt{\frac{EJ}{ml^4}}$).

In second approximation

$$\omega_{(2)}=10{,}93\sqrt{\frac{EJ}{ml^4}}.$$

The problem has an accurate solution

$$\omega=\frac{\pi^2}{\sqrt{1-\frac{I\pi^2}{ml^2}}}\sqrt{\frac{EJ}{ml^4}}=10{,}73\sqrt{\frac{EJ}{ml^4}};$$

$$\left(\frac{I}{ml^2}=\frac{1}{16}\frac{d^2}{l^2}=\frac{1}{64}\right).$$

From the calculation it is clear that first approximation gives a deviation of an order of 1.6%, and the inaccuracy of the second approximation (1.8%) is explained by error in the approximate integration. It can be removed by the selection of a greater number of sectors, however, a great accuracy in the calculation is not required.

4. Stability of Rods

The application of integral equations in problems of stability are especially effective, since for practical purpose there is required a seeking of only the minimum eigenvalue*.

We shall consider the stability of a rod of variable section under the effect of concentrated and distributed along the length.

The differential equation of the problem has the form

$$\frac{d^2}{dz^2}\left(EJ\frac{d^2y}{dz^2}(z)\right)+\frac{d}{dz}\left(P(z)\frac{dy}{dz}(z)\right)=0, \qquad (4.1)$$

where P(z) -- compressive force in section z;
EJ -- minimum strength of section to flexure.

Equation (4.1) is valid for any fastened ends of the rod.

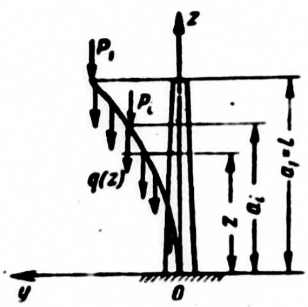

Fig. 17. Stability of Cantilever Rod.

We shall take as an example a cantilever rod (Fig. 17). Since a transverse force in section z = 1 is absent, then order of equation (4.1) can be lowered:

$$\frac{d}{dz}\left(EJ\frac{d^2y}{dz^2}\right)+P(z)\frac{dy}{dz}=0. \qquad (4.2)$$

*The application of boundary integral equations to problems of stability were for the first time given by Yu. V. Repman.

In the considered case

$$P(z)=\int_z^l q(z_1)\,dz_1+\sum_{i=1}^{n} S(z,a_i)P_i,$$

where P_i is the concentrated force in section $z=a_i$.

The single function $S(z,a_i)$ is determined now by equality

$$S(z,a_i)=\begin{cases}1 & z<a_i,\\ 0 & z\geqslant a_i.\end{cases}$$

If the stress P_i is directed for elongation, then into the calculation equation it should be introduced with a minus sign. In problems of stability, external loads contain as factors, parameters subject to determination.

Thus,

$$\left.\begin{aligned}&P(z)=\lambda_0 f_0(z)+\lambda_1 f_1(z)+\ \ldots\ +\lambda_n f_n(z),\\ &\lambda_0 f_0=\int_z^l q(z_1)\,dz_1,\quad \lambda_1 f_1=S(z,a_1)P_1.\end{aligned}\right\}\qquad(4.3)$$

For the concentrated forces as parameters λ_i usually there are taken the magnitudes P_i, i.e. $\lambda_i=P_i$ $(i=1,\ldots,n)$. Frequently as λ_i it is convenient to take the dimensionless parameters, for example $\lambda_i=\frac{P_i l^2}{EJ}$.

For concrete calculation one should note dependence between parameters λ_i so that expression (4.3) contains only one independent parameter λ (for example, $\lambda_0=\lambda,\ \lambda_1=0{,}5\lambda,\ \lambda_2=1{,}2\lambda$ et cetera).

. As chief unknown we shall take the value

$$\frac{dy}{dz}(z)=\varphi(z).$$

Equation (4.2) now will be written out as:

$$\frac{d}{dz}\left(EJ\frac{d\varphi}{dz}\right)=-\varphi(z)\left(\lambda_0 f_0(z)+\sum_{i=1}^{n}\lambda_i S(z,a_i)\right).$$

By integrating both sides of equality from z to 1 and considering the boundary condition

$$\frac{d\varphi}{dz}(l)=0,$$

we obtain

$$EJ\frac{d\varphi}{dz}=\lambda_0\int_z^l \varphi(z_1)f_0(z_1)\,dz_1+\sum_{i=1}^{n}\lambda_i S(z,a_i)\int_z^{a_i}\varphi(z_1)\,dz_1.\qquad(4.4)$$

We note that on the right hand-side of the equality is the expression of bending moment. It can be determined also directly from consideration of Fig. 17:

$$M(z)=\int_z^l q(z_1)\left[y(z_1)-y(z)\right]dz+\sum_{i=1}^{n} P_i S(z,a_i)\left[y(a_i)-y(z)\right];\qquad(4.5)$$

since

$$\lambda_0 f_0 = \int_z^l q(z_1)\, dz_1,$$

Then equations (4.4) and (4.5) in accuracy agree. After dividing both sides of equality by EJ and integrating then from 0 to z $(\varphi(0)=0)$, we obtain the integral equation

$$\varphi = \lambda K\varphi. \tag{4.6}$$

Operator
$$K\varphi = \nu_0 \int_0^z \frac{1}{EJ(z_1)} \int_{z_1}^l f_0(z_2)\, \varphi(z_2)\, dz_2\, dz_1 + \sum_{i=1}^n \nu_i B(z, a_i),$$

where

$$\nu_i = \frac{\lambda_i}{\lambda} \qquad (i=0, 1, \ldots, n),$$

$$B(z, a_i) = \begin{cases} \displaystyle\int_0^z \frac{1}{EJ(z)} \int_{z_1}^{a_i} \varphi(z_2)\, dz_2\, dz_1 & z < a_i, \\ \displaystyle\int_0^{a_i} \frac{1}{EJ(z)} \int_{z_1}^{a_i} \varphi(z_2)\, dz_2\, dz_1 & z \geqslant a_i. \end{cases}$$

One of the coefficients ν_i may be taken as arbitrary (for example, $\nu_0=1$).

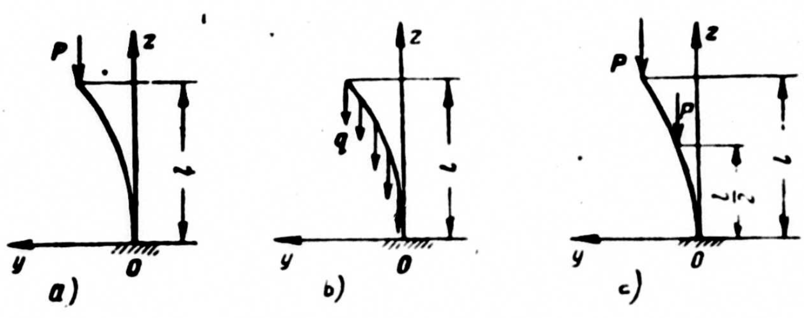

Fig. 18. Stability of Rod of Constant Section.

Expression for $B(z, a_i)$ it may be written out also in a simpler form

$$B(z, a_i) = \int_0^{z<a_i} \frac{1}{EJ(z)} \int_{z_1}^{a_i} \varphi(z_2)\, dz_2\, dz_1,$$

if it is agreed at $z \geqslant a_i$ to maintain the same value of the function $B(z, a_i)$. The integral equation (4.6) has a simple structure.

It is interesting to note that the use of differential equations would require the construction of solutions in each sector and a linking of the solutions with a calculation of boundary conditions.

Let us consider several particular cases.

Suppose we have a rod of constant section with force at end (Fig. 18,a).

For this case, equation (4.6) will be such:

or
$$\varphi(z) = \frac{P}{EJ}\int_0^z\int_{z_1}^l \varphi(z_2)\,dz_2\,dz_1$$

where
$$\varphi(\xi) = \lambda\int_0^\xi\int_{\xi_1}^1 \varphi(\xi_2)\,d\xi_2 d\xi_1,$$

$\lambda = \frac{Pl^2}{EJ}$; $\xi = \frac{z}{l}$. The accurate solution is $\lambda = 2,467$.

As the initial approximation we shall select

$$\varphi_{(0)} = \xi - \frac{1}{3}\xi^3, \tag{4.7}$$

satisfying main boundary conditions $\varphi(0) = 0$, $\varphi'(1) = 0$.

By the method of comparing ordinates*

$$\lambda_{(1)} = \left.\frac{\varphi_{(0)}}{K\varphi_{(0)}}\right|_{\xi=1} = 2{,}500\ (1{,}34).$$

By the method of minimum square deviation

$$\lambda_{(1)} = \frac{\int_0^1 \varphi_{(0)}K\varphi_{(0)}\,d\xi}{\int_0^1 (K\varphi_{(0)})^2\,d\xi} = 2{,}467\ (0{,}00).$$

In the following approximation by the method of comparing ordinates

$$\lambda_{(2)} = 2{,}471\ (0{,}16).$$

If even we select a rougher initial approximation

$$\varphi_{(0)} = \xi, \tag{4.8}$$

then by the method of comparing ordinates

$$\lambda_{(1)} = 3{,}000\ (21,6);\quad \lambda_{(2)} = 2{,}500\ (1,34);\quad \lambda_{(3)} = 2{,}471\ (0{,}16).$$

*In parentheses is shown the error in %.

If one were to apply approximate integrating by trapezodial rules, as is done in practical calculations, then by subdividing into ten equal sectors we obtain

$$\lambda_{(1)}=3{,}008;\quad \lambda_{(2)}=2{,}511;\quad \lambda_{(3)}=2{,}482.$$

By determining with the initial approximation (4.8) $\lambda_{(1)}$ by the method of minimum square deviation, we find

$$\lambda_{(1)}=2{,}470.$$

For a rod, loaded by a distributed load (Fig. 18, b), equation (4.6) will be such:

$$\varphi(z)=\frac{q}{EJ}\int_0^z\int_{z_1}^l (l-z_2)\,\varphi(z_2)\,dz_2\,dz_1, \qquad (4.9)$$

since the compressive force in section z

$$P(z)=\lambda_0 f_0(z)=q(l-z)$$

In converting to dimensionless form,

$$\varphi(\xi)=\lambda\int_0^\xi\int_{\xi_1}^1 (1-\xi_2)\,\varphi(\xi_2)\,d\xi_2\,d\xi_1,$$

where

$$\lambda=\frac{ql^3}{EJ}.$$

After taking a rough initial approximation in the form

$$\varphi_{(0)}=\xi,$$

we obtain by the method of minimal square deviation

$$\lambda_{(1)}=9.05,$$

$$\lambda_{(2)}=8.01,$$

$$\lambda_{(3)}=7.93$$

with an accurate value

$$\lambda = 7{,}837.$$

For a rod under the action of two concentrated forces (Fig.18b) from equation (4.6) we will have

$$\varphi(z)=\frac{P}{EJ}\left\{\int_0^z\int_{z_1}^l \varphi(z_2)\,dz_2\,dz_1+\int_0^{z<\frac{l}{2}}\int_{z_1}^{\frac{l}{2}} \varphi(z_2)\,dz_2\,dz_1\right\}. \qquad (4.10)$$

By setting up an initial approximation $\varphi_{(0)}=\xi$, we obtain by the method of comparing ordinates $\lambda_{(1)}=\dfrac{P_{(1)}l^2}{EJ}=2{,}67;\ \lambda_{(2)}=2{,}128.$

The accurate value $\lambda=2{,}068.$

We consider now the integral equations of stability of thin-walled rods of

constant section.

In the presence of distributed longitudinal and transverse stresses the problem is described by the following system of differential equations, obtained by V. Z. Vlasov:

$$EJ_y\xi^{IV} - [N(\xi' + a_y\theta')]' + (M_x\theta)'' = 0, \quad (4.11)$$

$$EJ_x\eta^{IV} - [N(\eta' - a_x\theta')]' + (M_y\theta)'' = 0, \quad (4.12)$$

$$EJ_\omega\theta^{IV} - GT\theta'' - [(r^2N + 2\beta_yM_x - 2\beta_xM_y)\theta']' + [q_x(e_x - a_x) + $$
$$+ q_y(e_y - a_y)]\theta - a_y(N\xi')' + a_x(N\eta')' + M_x\xi'' + M_y\eta'' = 0,$$

where ξ and η -- are components of displacement of center of flexure along the principal axes x and y arising with the loss of stability;

θ -- is complementary angle of rotation of section during loss of stability;

N, M_x, M_y -- are the normal (tensile) force and the bending moments in the section of rod under action of external load;

J_ω -- is sectorial moment of inertia;

T -- is the geometric rigidity to torsion;

q_x and q_y -- are components of transverse distributed load;

e_x and e_y -- are coordinates of point of application of distributed load in plane of the section;

a_x and a_y -- are coordinates of center of bend;

r, β_x and β_y -- are geometric characteristics of section.

In composing the boundary integral equation as the chief unknowns it is expedient to take

$$\frac{d^2\xi}{dz^2} = \varphi; \quad \frac{d^2\eta}{dz^2} = \psi; \quad \frac{d\theta}{dz} = \vartheta.$$

We now consider for example, cantilever rod (Fig. 19) with free upper and rigidly fixed lower sections. In this case, we shall have

$$\xi'(z)=\int_0^z \varphi(z_1)\,dz_1; \qquad \xi(z)=\int_0^z\int_0^{z_1} \varphi(z_2)\,dz_2\,dz_1;$$

$$\eta'(z)=\int_0^z \psi(z_1)\,dz_1; \qquad \eta(z)=\int_0^z\int_0^{z_1} \psi(z_2)\,dz_2\,dz_1;$$

$$\theta(z)=\int_0^z \vartheta(z_1)\,dz_1.$$

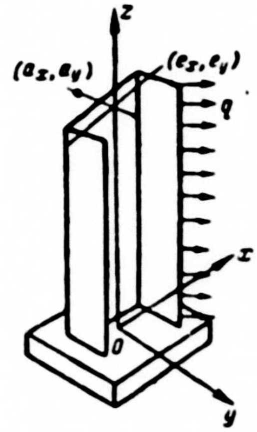

Fig. 19. General case of stability of thin-walled rod.

We shall write out equations (4.11)--(4.13) in the following form;

$$\varphi''=\frac{1}{EJ_y}\left[N\left(\int_0^z \varphi(z_1)\,dz_1+a_y\vartheta\right)\right]'-\frac{1}{EJ_y}\left(M_x\int_0^z \vartheta(z_1)\,dz_1\right)'', \qquad (4.14)$$

$$\psi''=\frac{1}{EJ_x}\left[N\left(\int_0^z \psi(z_1)\,dz_1-a_x\vartheta\right)\right]'-\frac{1}{EJ_x}\left(M_y\int_0^z \vartheta(z_1)\,dz_1\right)'', \qquad (4.15)$$

$$\vartheta'''=\frac{1}{EJ_\omega}\left[a_y\left(N\int_0^z \varphi(z_1)\,dz_1\right)'-a_x\left(N\int_0^z \psi(z_1)\,dz_1\right)'+r^2(N\vartheta)'\right]-$$
$$-\frac{1}{EJ_\omega}\Big[M_x\varphi+M_y\psi-((2\beta_yM_x-2\beta_xM_y)\,\vartheta)'+$$
$$+(q_x(e_x-a_x)+q_y(e_y-a_y))\int_0^z \vartheta(z_1)\,dz_1\Big]+\frac{GT}{EJ_\omega}\vartheta'. \qquad (4.16)$$

In integrating both sides of equalities (4.14) and (4.15) twice from z to l and by considering boundary conditions at z = l we obtain

$$\varphi(z)=-\frac{1}{EJ_y}\int_z^l N(z_1)\left(\int_0^{z_1} \varphi(z_2)\,dz_2+a_y\vartheta(z_1)\right)dz_1+$$
$$+\frac{1}{EJ_y}\left(M_x\int_0^z \vartheta(z_1)\,dz_1\right)\Bigg|_z^l, \qquad (4.17)$$

$$\psi(z)=-\frac{1}{EJ_x}\int_z^l N(z_1)\left(\int_0^{z_1} \psi(z_2)\,dz_2-a_x\vartheta(z_1)\right)dz_1+$$

$$+\frac{1}{EJ_x}\left(M_y\int_0^z \vartheta(z_1)\,dz_1\right)\Big|_z^l, \tag{4.18}$$

$$\vartheta' = -\frac{1}{EJ_\omega}\int_z^l N(z_1)\left(a_y\int_0^{z_1}\varphi(z_2)\,dz_2 - a_x\int_0^{z_1}\psi(z_2)\,dz_2 + r^2\vartheta(z_1)\right)dz_1 -$$

$$-\frac{1}{EJ_\omega}\left[-\int_z^l\int_0^{z_1}\left(M_x\varphi(z_2)+M_y\psi(z_2)\right)dz_2\,dz_1 + \int_z^l(2\beta_y M_x -\right.$$

$$\left.-2\beta_x M_y)\,\vartheta(z_1)\,dz_1 + \int_z^l\left(q_x(e_x-a_x)+q_y(e_y-a_y)\right)\int_0^{z_1}\vartheta(z_2)\,dz_2\,dz_1\right] -$$

$$-\frac{GT}{EJ_\omega}\int_z^l \vartheta(z_1)\,dz_1. \tag{4.19}$$

By integrating equality (4.19) from 0 to z, we find

$$\vartheta = -\frac{1}{EJ_\omega}\int_0^z\int_{z_1}^l N(z_2)\left(a_y\int_0^{z_2}\varphi(z_3)\,dz_3 - a_x\int_0^{z_2}\psi(z_3)\,dz_3 +\right.$$

$$\left.+r^2\vartheta(z_2)\right)dz_2\,dz_1 - \frac{1}{EJ_\omega}\left[-\int_0^z\int_{z_1}^l\int_0^{z_2}\left(M_x\varphi(z_3)+M_y\psi(z_3)\right)dz_3\,dz_2\,dz_1 +\right.$$

$$+\int_0^z\int_{z_1}^l(2\beta_y M_x - 2\beta_x M_y)\,\vartheta(z_2)\,dz_2\,dz_1 + \int_0^z\int_{z_1}^l\left(q_x(e_x-a_x)+\right.$$

$$\left.+q_y(e_y-a_y)\right)\int_0^{z_2}\vartheta(z_3)\,dz_3\,dz_2\,dz_1 - \frac{GT}{EJ_\omega}\int_0^z\int_{z_1}^l\vartheta(z_2)\,dz_2\,dz_1. \tag{4.20}$$

In practical problems , external loads usually can be presented in the llowing form:

$$N(z) = -\lambda_1 n_1(z) + \lambda_2 n_2(z). \tag{4.21}$$

$$M_x(z) = \lambda_3 m_{1x}(z) + \lambda_4 m_{2x}(z), \tag{4.22}$$

$$M_y(z) = \lambda_3 m_{1y}(z) + \lambda_4 m_{2y}(z), \tag{4.23}$$

$$q_x(e_x-a_x) + q_y(e_y-a_y) = \lambda_4 t(z), \tag{4.24}$$

re parameters λ_j will be subject to determination.

Equations (4.17), (4.18) and (4.20) are written in matrix form

$$\begin{bmatrix}\varphi\\ \psi\\ \vartheta\end{bmatrix} - \sum_{j=1}^{4}\lambda_j\begin{bmatrix}K_{j11} & K_{j12} & K_{j13}\\ K_{j21} & K_{j22} & K_{j23}\\ K_{j31} & K_{j32} & K_{j33}\end{bmatrix}\begin{bmatrix}\varphi\\ \psi\\ \vartheta\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & K_{033}\end{bmatrix}\begin{bmatrix}\varphi\\ \psi\\ \vartheta\end{bmatrix}, \tag{4.25}$$

where values of the operators are readily established after a comparison of corresponding equations.

If one were to introduce the matrix-column $[\Phi]$, then equation (4.25) can be written out even more briefly:

$$[\Phi]=\sum_{j=1}^{4}\lambda_j[K_j][\Phi]+[K_0][\Phi]. \qquad (4.26)$$

Equation (4.26) is a five-parametric boundary matrix integral equation.

For a concrete problem there should be known the relationship between parameters of load $\frac{\lambda_j}{\lambda}=\nu_j$ then we obtain the two parametric equation

$$[\Phi]=\lambda[K][\Phi]+[K_0][\Phi], \qquad (4.27)$$

where

$$[K]=\sum_{j=1}^{4}\nu_j[K_j].$$

Two-parametric integral equation already has been encountered in problem on vibration of rod in a field of centrifugal forces.

If the equation $[\Phi]=\mu[K_0][\Phi]$ has the eigenvalue $[\mu_1]<1$, then for the solution of equation (4.27) there can be used the method of simple iteration:

$$[\Phi_{(i)}]=\lambda_{(i)}[K][\Phi_{(i-1)}]+[K_0][\Phi_{(i-1)}].$$

The value $\lambda_{(i)}$ is sought, for example, by means of comparison of the maximum values.

$$\|[\Phi]\|=\sqrt{\varphi^2+\psi^2+\vartheta^2}$$

for the (i-1)-th and i-th approximation (See Chap. 3, Sec. 3):

$$\|[\Phi_{(i)}]\|=\|[\Phi_{(i-1)}]\|_{s=s_{mi}}.$$

Then we obtain

$$\lambda_{(i)}=\left.\frac{\|[\Phi_{(i-1)}]-[K_0][\Phi_{(i-1)}]\|}{\|[K][\Phi_{(i-1)}]\|}\right|_{s=s_{mi}}. \qquad (4.28)$$

If $|M_1|>1$, then it is possible to apply method of similar iteration.

For the equation $[y]=\lambda[K][v]+[K_0][y]$

we have

$$y_{(i+1),\nu}=\lambda_{(i)}\frac{\sum_{j=1}^{n}K_{\nu j}y_{(i),j}}{y_{(i),\nu}-\sum_{j=1}^{n}K_{0\nu j}y_{(i),j}}\,y_{(i),\nu},$$

$$\nu=1,2,\ldots,n.$$

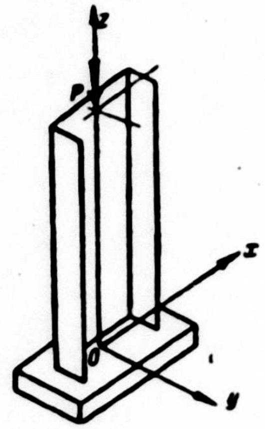

Fig. 20. Stability of Thin-walled Rod Under Action of Longitudinal Force.

We shall give an example. Suppose it is required to determine the value of force P, causing the loss of stability of rod (Fig. 20). The force is applied at the center of gravity of section.

In equality (4.21) we shall take

$$n_1(z) = 1; \quad \lambda_1 = \lambda = \frac{Pl^2}{EJ_x}.$$

Then from equation (4.25) there ensues (the subscript $i = 1$ we omit)

$$\varphi = \lambda\,(K_{11}\varphi + K_{12}\psi + K_{13}\vartheta), \tag{4.29}$$

$$\psi = \lambda\,(K_{21}\varphi + K_{22}\psi + K_{23}\vartheta), \tag{4.30}$$

$$\vartheta = \lambda\,(K_{31}\varphi + K_{32}\psi + K_{33}\vartheta) + K_{033}\vartheta, \tag{4.31}$$

where

$$K_{11}\varphi = \frac{EJ_x}{EJ_y}\int_{\zeta}^{1}\int_{0}^{\zeta_1}\varphi(\zeta_2)\,d\zeta_2\,d\zeta_1; \quad K_{12}\psi = 0; \quad K_{13}\vartheta = \frac{EJ_x}{EJ_y}\frac{a_y}{l}\int_{\zeta}^{1}\vartheta(\zeta_1)\,d\zeta_1;$$

$$K_{21}\varphi = 0; \quad K_{22}\psi = \int_{\zeta}^{1}\int_{0}^{\zeta_1}\psi(\zeta_2)\,d\zeta_2\,d\zeta_1; \quad K_{23}\vartheta = -\frac{a_x}{l}\int_{\zeta}^{1}\vartheta(\zeta_1)\,d\zeta_1;$$

$$K_{31}\varphi = \frac{EJ_x a_y l}{EJ_\omega}\int_{0}^{\zeta}\int_{\zeta_1}^{1}\int_{0}^{\zeta_2}\varphi(\zeta_3)\,d\zeta_3\,d\zeta_2\,d\zeta_1; \quad K_{32}\psi = -\frac{EJ_x a_x l}{EJ_\omega}\int_{0}^{\zeta}\int_{\zeta_1}^{1}\int_{0}^{\zeta_2}\psi(\zeta_3)\,d\zeta_3\,d\zeta_2\,d\zeta_1;$$

$$K_{33}\vartheta = \frac{EJ_x}{EJ_\omega}r^2\int_{0}^{\zeta}\int_{\zeta_1}^{1}\vartheta(\zeta_2)\,d\zeta_2\,d\zeta_1; \quad K_{033}\vartheta = -\frac{GTl^2}{EJ_\omega}\int_{0}^{\zeta}\int_{\zeta_1}^{1}\vartheta(\zeta_2)\,d\zeta_2\,d\zeta_1.$$

The calculation of the presented operators is comparatively simple, since they contain only three different integral expressions. If section of rod possesses an axis of symmetry, (for example, the y axis), then center of rigidity is located on this axis $(a_x = 0)$. Then equation (4.30) becomes independent

$$\psi = \lambda\int_{\zeta}^{1}\int_{0}^{\zeta_1}\psi(\zeta_2)\,d\zeta_2\,d\zeta_1, \tag{4.32}$$

and the two other will form a system.

The minimum eigenvalue of equation (4.32) corresponds to the Euler force.

$$P_1 = \frac{\pi^2 EJ_x}{4l^2}. \tag{4.33}$$

In the solution of equations (4.27) and (4.31) by the method of successive approximations as the initial approximations it is possible to take

$$\varphi_{(0)}(\zeta) = 1 - \zeta, \quad \vartheta_{(0)}(\zeta) = \zeta,$$

what satisfies conditions $\varphi(1) = 0, \ \vartheta(0) = 0.$

Further, we shall have

$$\varphi_{(1)} = \lambda_{(1)} \left(\frac{EJ_x}{EJ_y} \int_{\zeta}^{1} \int_{0}^{\zeta_1} \varphi_{(0)}(\zeta_2)\, d\zeta_2\, d\zeta_1 + \frac{EJ_x}{EJ_y} \frac{a_y}{l} \int_{\zeta}^{1} \vartheta_{(0)}(\zeta_1)\, d\zeta_1 \right),$$

$$\vartheta_{(1)} = \lambda_{(1)} \left(\frac{EJ_x a_y l}{EJ_\omega} \int_{0}^{\zeta} \int_{\zeta_1}^{1} \int_{0}^{\zeta_2} \varphi_{(0)}(\zeta_3)\, d\zeta_3\, d\zeta_2\, d\zeta_1 + \frac{EJ_x}{EJ_\omega} r^2 \int_{0}^{\zeta} \int_{\zeta_1}^{1} \vartheta_{(0)}(\zeta_2)\, d\zeta_2\, d\zeta_1 \right) - \tag{4.34}$$

$$- \frac{GTl^2}{EJ_\omega} \int_{0}^{\zeta} \int_{\zeta_1}^{1} \vartheta(\zeta_2)\, d\zeta_2\, d\zeta_1.$$

In a similar manner there can be considered also more complex questions on the stability of rods.

5. Extension and Flexure of Round Plates (Disks)

The indicated problem by virtue of its practical importance for calculation of disks in turbomachines has been investigated by different methods. However, also in it, the application of integral equations* makes it possible to construct one of the most effective solutions.

We shall consider an axially symmetric extension of disk under action of centrifugal forces and nonuniform heating (Fig. 21). Parameters of elasticity of material of disk (E and μ) are assumed depending on radius.

Problem is described by a differential second order equation relative to the

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*R. S. Kinasoshvili, Calculation for Strength of Turbomachine Disks. Defense Ministry Publ. House, Moscow, 1954; I. A. Birger, Integral Methods of Calculation of Disk, Collection MAP No. 6, Defense Ministry Publ. House, Moscow, 1950.

radial displacement $u(r)$:

$$\left.\begin{aligned}&\frac{d^2u}{dr^2}+\frac{d}{dr}(\ln H)\frac{du}{dr}+\left[\frac{\mu}{r}\frac{d}{dr}(\ln H)+\frac{d}{dr}\left(\frac{\mu}{r}\right)-\frac{1}{r^2}\right]u=f(r);\\&f(r)=(1+\mu)\alpha t\frac{d}{dr}(\ln H)+\frac{d}{dr}\left[(1+\mu)\alpha t\right]-\frac{(1+\mu)\alpha t}{r}-\\&\qquad-q(r)\frac{1-\mu^2}{E},\end{aligned}\right| \tag{5.1}$$

where

$H=\frac{rhE}{1-\mu^2}$; αt – is the temperature deformation; q(r) is the intensity of body force (for case of action of centrifugal forces

$$q(r)=\rho\omega^2 r, \tag{5.2}$$

here ρ is the density of material of disk; ω angular velocity of rotation), or system of two first order differential equations (equation of equilibrium and equation of congruence)

$$\frac{d}{dr}(\sigma_r h)-\frac{(\sigma_\theta-\sigma_r)h}{r}+\rho\omega^2 rh=0, \tag{5.3}$$

$$\frac{1+\mu}{rE}(\sigma_r-\sigma_\theta)=\frac{d}{dr}\left[\frac{1}{E}(\sigma_\theta-\mu\sigma_r)+\alpha t\right], \tag{5.4}$$

where σ_r and σ_θ are the radial and circumferential stresses.

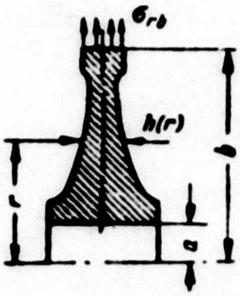

Fig. 21. Extension of Disk.

It is possible to construct different integral equations of the problem, where for practical application it is important, so that equation does not contain derivatives of the initial parameters of disk $(h, E, \alpha t)$.

By integrating both sides of equations (5.3) and (5.4) from a and to r, we obtain

$$\sigma_r=\frac{1}{h}\left[\int_a^r\frac{(\sigma_\theta-\sigma_r)h}{r_1}dr_1-\rho\omega^2\int_a^r r_1 h\,dr_1+a\sigma_{ra}\right], \tag{5.5}$$

$$\begin{aligned}\sigma_\theta-\sigma_r=&-(1-\mu)\sigma_r-E\int_a^r\frac{1+\mu}{r_1E}(\sigma_\theta-\sigma_r)\,dr_1-\\&-E(\alpha t-\alpha_a t_a)+\frac{E}{E_a}(\sigma_{\theta a}-\mu\sigma_{ra}).\end{aligned} \tag{5.6}$$

The subscript a in these dependence indicates that the value of the parameter refers to r = a. By introducing σ_r from relationship (5.5) into equality (5.6), we obtain normal integral equation relative to

$$y(r)=\sigma_\theta(r)-\sigma_r(r).$$

This equation has the form

$$y(r)=-\frac{1-\mu}{h}\int_a^r \frac{h}{r_1}y(r_1)\,dr_1-E\int_a^r \frac{1+\mu}{r_1E}y(r_1)\,dr_1+$$

$$+\frac{1-\mu}{h}\rho\omega^2\int_a^r r_1h\,dr_1-E(\alpha t-\alpha_a t_a)+\frac{E}{E_a}(\sigma_{\theta a}-\mu_a\sigma_{ra})- \qquad (5.7)$$

$$-\frac{1-\mu}{h}h_a\sigma_{ra}.$$

We shall write it out in following form:

$$y(r)=Q_1(r)\int_a^r q_1(r_1)y(r_1)\,dr_1+Q_2(r)\int_a^r q_2(r_1)y(r_1)\,dr_1+ \qquad (5.8)$$

$$+f+\sigma_{\theta a}f_1,$$

where

$$Q_1(r)=-\frac{1-\mu}{h},\quad q_1(r)=\frac{h}{r},\quad Q_2(r)=-E;\quad q_2(r)=\frac{1+\mu}{rE};$$

$$f=\frac{1-\mu}{h}\rho\omega^2\int_a^r r_1h\,dr_1-E(\alpha t-\alpha_a t_a).$$

For a solid disk $(\sigma_{ra}=\sigma_{\theta a})$

$$f_1=\frac{E}{E_a}(1-\mu_a)-\frac{1-\mu}{h}h_a.$$

For a disk with an aperture $(\sigma_{ra}=0)$

$$f_1=\frac{E}{E_a}.$$

The value $\sigma_{\theta a}$ is determined from the boundary condition

$$\sigma_r(b)=\sigma_{rb}.$$

For the solution of equation (5.8) it is expedient to apply method of linear approximation (See Chap. 2, Sec. 4), which turns out to be in the given case more effective, than the method of successive approximations.

The problem can be reduced to the normal integral equation

$$\sigma_\theta(r) = \overset{\prime\prime}{N}\sigma_\theta + F_0(r) + \sigma_{ra}F_1(r) + \sigma_{\theta a}F_2(r),$$

$$N\sigma_\theta = \frac{E(r)G(r)}{r}\int_a^r h(r_1)\sigma_\theta(r_1)\,dr_1 -$$

$$-\frac{E(r)}{r}\int_a^r h(r_1)G(r_1)\sigma_\theta(r_1)\,dr_1, \qquad (5.9)$$

where

$$G(r) = \int_a^r \frac{dr_1}{r_1 h(r_1)E(r_1)} + \frac{\mu(r)}{h(r)E(r)}.$$

Calculation of disk in elasto-plastic strains can be made on the basis of equations, valid for an elastic disk with variable parameters of elasticity*. This remark refers also to the calculation of disk for creep on the basis of theory of aging. Problem about symmetric flexure of disk (Fig. 22) has much in common with problem on extension.

The differential equation of flexure of disk with a calculating of nonuniform heating through thickness of disk and of forces in middle plane, has the form

$$\frac{d^2\varphi}{dr^2} + \frac{d}{dr}(\ln rD) + \left[\frac{\mu}{r}\frac{d}{dr}(\ln rD) + \frac{d}{dr}\left(\frac{\mu}{r}\right) - \right.$$

$$\left. -\frac{1}{r^2} - \frac{N_r}{D}\right]\varphi = f(r), \qquad (5.10)$$

$$f(r) = \frac{\alpha\Delta t}{h}(1+\mu)\frac{d}{dr}(\ln rD) + \frac{d}{dr}\left[\frac{\alpha\Delta t}{h}(1+\mu)\right] -$$

$$-\frac{\alpha}{r}\Delta t(r)(1+\mu) + \frac{1}{rD}\int_a^r q(r_1)\,r_1\,dr_1 - \frac{a}{rD}(Q_a + N_{ra}\varphi_a).$$

In this equation

$\varphi(r)$ -- angle of rotation of the normal to middle plane of disk;

$D(r) = \frac{Eh^3}{12(1-\mu^2)}$ -- cylindrical rigidity of disk on the radius r;

$N_r = \sigma_r h$ -- is the tensile radial force in section r;

*I. A. Birger, Certain General Methods of Solving of Problems in Theory of Plasticity, "Applied Mathematics and Mechanics, Vol. 15, No. 6, 1951.

$q(r)$ -- is the distributed load, perpendicular to middle plane of disk;

$\Delta t(r)$ -- is the difference of temperature through thickness of disk.

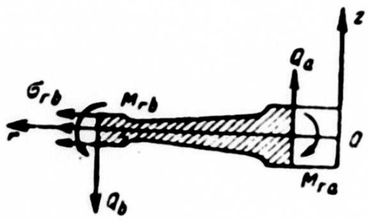

Fig. 22. Flexure of Disk.

Temperature at the point, at a distance $\underset{\sim}{z}$ from the middle plane, is assumed equal

$$t(r, z) = \frac{\Delta t(r)}{h(r)} z.$$

By using the equation of equilibrium and equation of congruence relative to the bending moments, we arrive at the normal integral equation

$$M_r(r) = NM_r + F(r), \tag{5.11}$$

where

$$NM_r = \int_a^r \left(\frac{D(r_1)(1-\mu^2)}{r_1^{2+\mu}} + \frac{N_r(r_1)}{r_1^{\mu}} \right) \int_a^{r_1} \frac{r_2^{\mu}}{D(r_2)} M_r(r_2)\, dr_2\, dr_1 -$$

$$-\int_a^r \frac{1-\mu}{r_1} M_r(r_1)\, dr_1;$$

$$F(r) = \int_a^r \left[(f(r) - f(r_1))(1+\mu) r_1^{\mu} - \frac{D(r_1)(1-\mu^2)}{r_1} \right] \frac{\Delta t(r_1)\alpha(r_1)}{h(r_1)} dr_1 -$$

$$-\int_a^r \tilde{Q}(r_1)\, dr_1 + \varphi_a \left[a^{\mu} f(r) - a N_{ra} \ln \frac{r}{a} \right] + M_{ra};$$

$$f(r) = \int_a^r \left(\frac{D(r_1)(1-\mu^2)}{r_1^{2+\mu}} + \frac{N_r(r_1)}{r_1^{\mu}} \right) dr_1;$$

$$\tilde{Q}(r) = -\frac{1}{r} \int_a^r q(r_1)\, r_1\, dr_1 + \frac{1}{r} a Q_a.$$

We now turn to a consideration of the general case of flexure of disk (round plate).

The differential equation of problem has the form

$$D\nabla^4 w + \frac{dD}{dr}\left[2\frac{\partial^3 w}{\partial r^3} + \frac{2+\mu}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} - \frac{3}{r^3}\frac{\partial^2 w}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial^3 w}{\partial r\, \partial \theta^2} \right] + \tag{5.12}$$

$$+ \frac{d^2 D}{dr^2}\left[\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r} + \frac{\mu}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right] - \frac{1}{r}\frac{\partial}{\partial r}\left(\sigma_r h r \frac{\partial w}{\partial r} \right) -$$

$$- \frac{\sigma_\theta h}{r^2}\frac{\partial^2 w}{\partial \theta^2} = -q - \nabla^2 T,$$

where $w(r, \theta)$ is the flexure of middle plane of disk.

The temperature of the point of disk

$$t(r, \theta, z) = \frac{\Delta t(r, \theta)}{h(r)} z;$$

the function

$$T(r, \theta) = (1+\mu)\alpha \frac{\Delta t}{h} D.$$

The radial σ_r and circumferential σ_θ stresses in middle plane of disk possess axial symmetry.

For composing the integral equation of problem, we shall use the equation of equilibrium in integral form

$$M_r = -\frac{1}{r}\int_r^b \left(M_\theta + 2\frac{\partial M_{r\theta}}{\partial\theta}\right) dr_1 + \frac{1}{r}\int_r^b\int_{r_1}^b \frac{1}{r_2}\left(2\frac{\partial M_{r\theta}}{\partial\theta} - \frac{\partial^2 M_\theta}{\partial\theta^2}\right) dr_2\, dr_1 + \tag{5.13}$$
$$+\frac{1}{r}\int_r^b\int_{r_1}^b q r_2\, dr_2\, dr_1 + V_{rb}\frac{b}{r}(b-r) + \frac{b}{r} M_{rb},$$

where M_r, M_θ and $M_{r\theta}$ are the radial, circumferential moments and torques per unit of length; V_{rb}- is the transverse force on the contour $r=b$.

We shall consider, as example the case, when

$$q = q(r)\cos\theta. \tag{5.14}$$

Forces are absent in middle plane of disk, and the heating is nonuniform.

In accordance with equality (5.14)

$$w(r, \theta) = w(r)\cos\theta.$$

By introducing values M_r, M_θ and $M_{r\theta}$, expressed by the derivatives w, into equation (5.13), we shall obtain a boundary integral equation relative to

$$\frac{d^2 w(r)}{dr^2} = \psi(r):$$

$$\psi(r) = -\frac{\mu}{r^2}\int_a^r r_1\psi(r_1)\, dr_1 - \frac{1}{D(r)}\int_r^b \frac{D(r_1)}{r_1} \times$$
$$\times\left[\mu\psi(r_1) + \frac{3-2\mu}{r_1^2}\int_a^{r_1} r_2\psi(r_2)\, dr_2\right] dr_1 - \frac{1}{rD(r)}\int_r^b\int_{r_1}^b q(r_2)\, r_2\, dr_2\, dr_1 -$$
$$-\frac{b}{rD(r)} V_{rb}(b)(b-r) - \frac{b}{rD(r)} M_{rb}(b) +$$
$$+\left(w(a) - a\frac{dw}{dr}(a)\right)\left(\frac{\mu}{r^2} + \frac{3-2\mu}{D(r)}\int_r^b \frac{D(r_1)}{r_1^3}\, dr_1\right). \tag{5.15}$$

If disk (round plate) has its external contour free, and the internal fixed, then the magnitude V_{rb} and M_{rb} are given,

$$w(a) = 0; \quad \frac{dw}{dr}(a) = 0.$$

Equation (5.15)

$$\psi = K\psi + F$$

is solved by simple iteration, at $\|K\| < 1$ and by similar iteration at $K \geqslant 1$.

In a number of cases, already the initial approximation

$$\psi_{(0)} = F$$

gives a result with an accuracy of an order 10 to 15%.

Equations, analogous to (5.15), can be composed also for more general cases of loading.

6. Symmetric Deformation of Shells of Rotation

The problem has numerous applications in structural engineering and machine-building.

The solution of problem by means of finding accurate solutions of corresponding differential equations encounters great mathematical difficulties, especially for shells of variable thickness. In connection with this it is of interest to establish integral equations of an axially symmetric deformation of shells of rotation and to apply approximate methods of their solution.

The scheme of the shell is shown in Fig. 23.

The temperature of the material of shell is assumed linearly variable by thickness

$$t = t_0 + \frac{\Delta t}{h} z,$$

where t_0 is the temperature of points of middle surface;

Δt is the temperature drop through thickness of wall.

Ordinary variables are used: angle of rotation of normal to middle surface

$$\vartheta = \frac{u}{R_1} + \frac{dw}{ds} \tag{6.1}$$

and the magnitude $\eta = R_2 Q,$ (6.2)

where Q -- transverse force in section.

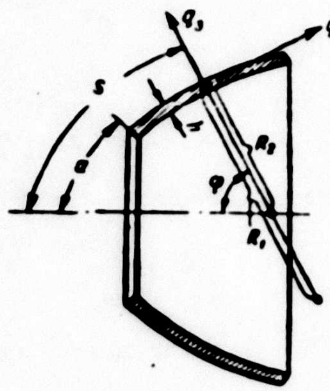

Fig. 23. For Calculating Shell of Rotation.

Relative to these variables the following system of differential equations is obtained

$$\frac{d}{ds}\left(\frac{R_2}{Eh}\frac{d\eta}{ds}\right)+\frac{\cot\varphi}{Eh}\left(\frac{R_2}{R_1}+\mu\right)\frac{d\eta}{ds}-\frac{d}{ds}\left(\frac{\mu\cot\varphi}{Eh}\eta\right)-$$
$$-\left(1+\mu\frac{R_2}{R_1}\right)\frac{\cot^2\varphi}{R_2Eh}\eta+F_q(s)+F_t(s)=\vartheta, \qquad (6.3)$$

$$\frac{1}{\sin\varphi}\frac{d}{ds}\left[DR_2\left(\frac{d\vartheta}{ds}\sin\varphi+\mu\frac{\vartheta}{R_2}\cos\varphi\right)\right]- \qquad (6.4)$$
$$-D\left(\frac{\vartheta}{R_2}\cos\varphi+\mu\frac{d\vartheta}{ds}\sin\varphi\right)\frac{\cos\varphi}{\sin^2\varphi}+\Phi_t(s)=-\eta.$$

In these equations E and μ are parameters of elasticity, variable along the arc of meridan s; $D=\frac{Eh^3}{12(1-\mu^2)}$ is the cylindrical rigidity. The functions $F_q(s)$, $F_t(s)$ and $\Phi_t(s)$ are determined by the equalities

$$F_q(s)=-\frac{d}{ds}\left[\frac{R_2}{2\pi\cos\varphi Eh}\frac{d}{ds}\left(\frac{P}{\sin\varphi}\right)-\mu\frac{P}{2\pi Eh\sin^2\varphi}\right]-$$
$$-\frac{1}{2\pi Eh\sin\varphi}\left(\frac{R_2}{R_1}+\mu\right)\frac{d}{ds}\left(\frac{P}{\sin\varphi}\right)+\frac{\cot\varphi}{2\pi EhR_2\sin^2\varphi}\left(1+\mu\frac{R_2}{R_1}\right)P-$$
$$-\frac{d}{ds}\left(q_1\frac{R_2^2\tan\varphi}{Eh}\right)-\left(\frac{R_2}{R_1}+\mu\right)\frac{R_2}{Eh}q_1, \qquad (6.5)$$

$$F_t(s)=\left(1-\frac{R_2}{R_1}\right)\cot\varphi\,\alpha t_0-\frac{d}{ds}(R_2\alpha t_0), \qquad (6.6)$$

$$\Phi_t(s)=D(1+\mu)\frac{\alpha\Delta t}{h}\cot\varphi-\frac{1}{\sin\varphi}\frac{d}{ds}\left[R_2\sin\varphi D(1+\mu)\frac{\alpha\Delta t}{h}\right]. \qquad (6.7)$$

where P is the resultant of all forces (concentrated and distributed) applied to intercepted part of shell. In composing the integral equations, we shall select as main unknown functions

$$\left.\begin{aligned}\eta^{(1)}(s)&=\frac{d\eta}{ds},\\ \vartheta^{(1)}(s)&=\frac{d\vartheta}{ds},\end{aligned}\right\} \qquad (6.8)$$

This makes it possible in subsequent calculations to avoid differentiation of initial geometric and elastic parameters, which essentially lower the accuracy of

the calculation.

Further, one should consider the equality

$$\left.\begin{aligned}\eta(s)&=\int_a^s\eta^{(1)}(\xi)\,d\xi+\eta(a),\\ \vartheta(s)&=\int_a^s\vartheta^{(1)}(\xi)\,d\xi+\vartheta(a).\end{aligned}\right\}\qquad(6.9)$$

By introducing the values (6.8) and (6.9) into equations (6.3) and (6.4) and by integrating both sides of equality from a to s, we will obtain a system of normal integral equations:

$$\left.\begin{aligned}\eta^{(1)}&=N_{11}\eta^{(1)}+N_{12}\vartheta^{(1)}+\eta(a)f_{11}+\eta^{(1)}(a)f_{12}+\\&\quad+\vartheta(a)f_{13}+\vartheta^{(1)}(a)f_{14}+f_1;\\ \vartheta^{(1)}&=N_{21}\eta^{(1)}+N_{22}\vartheta^{(1)}+\eta(a)f_{21}+\eta^{(1)}(a)f_{22}+\\&\quad+\vartheta(a)f_{23}+\vartheta^{(1)}(a)f_{24}+f_2,\end{aligned}\right|\qquad(6.10)$$

where

$$N_{11}\eta^{(1)}=\frac{Eh}{R_2}\left[\int_a^s\left(1+\mu\frac{R_2}{R_1}\right)\frac{\cot^2\varphi}{R_2Eh}\int_a^\xi\eta^{(1)}(\xi_1)\,d\xi_1\,d\xi+\right.$$

$$\left.+\frac{\mu\cot\varphi}{Eh}\int_a^s\eta^{(1)}(\xi)\,d\xi-\int_a^s\frac{\cot\varphi}{Eh}\left(\frac{R_2}{R_1}+\mu\right)\eta^{(1)}(\xi)\,d\xi\right],$$

$$N_{12}\vartheta^{(1)}=\frac{Eh}{R_2}\int_a^s\int_a^\xi\vartheta^{(1)}(\xi_1)\,d\xi_1\,d\xi,$$

$$N_{21}\eta^{(1)}=-\frac{1}{DR_2\sin\varphi}\int_a^s\sin\varphi\int_a^\xi\eta^{(1)}(\xi_1)\,d\xi_1\,d\xi,$$

$$N_{22}\vartheta^{(1)}=\frac{1}{DR_2\sin\varphi}\left[\int_a^s\left(\frac{\cos\varphi}{R_2}\int_a^\xi\vartheta^{(1)}(\xi_1)\,d\xi_1+\mu\sin\varphi\,\vartheta^{(1)}(\xi)\right)D\cot\varphi\,d\xi-\right.$$

$$\left.-D\mu\cos\varphi\int_a^s\vartheta^{(1)}(\xi)\,d\xi\right].$$

The functions f_{ij}, entering into equation (6.10), will be determined by the dependencies

$$f_{11}=\frac{Eh}{R_2}\left(\int_a^s\left(1+\mu\frac{R_2}{R_1}\right)\frac{\cot^2\varphi}{R_2Eh}\,d\xi+\frac{\mu\cot\varphi}{Eh}-\frac{\mu_a\cot\varphi_a}{E_ah_a}\right),$$

$$f_{12}=\frac{R_{2a}}{E_ah_a}\frac{Eh}{R_2},\quad f_{13}=\frac{Eh}{R_2}(s-a),\quad f_{14}=0,$$

$$f_{21}=-\frac{1}{DR_2\sin\varphi}\int_a^s\sin\varphi\,ds,\quad f_{22}=0,$$

$$f_{23} = \frac{1}{DR_2 \sin\varphi}\left(\int_a^s \frac{D}{R_2}\cos\varphi \cot\varphi \, d\xi - D\mu\cos\varphi + D_a\mu_a\cos\varphi_a\right).$$

$$f_{24} = \frac{D_a}{D}\frac{R_{2a}\sin\varphi_a}{R_2\sin\varphi}, \quad f_1 = -\frac{Eh}{R_2}\left(\int_a^s F_q(\xi)\,d\xi + \int_a^s F_t(\xi)\,d\xi\right).$$

$$f_2 = -\frac{1}{DR_2\sin\varphi}\int_a^s \Phi_t(\xi)\sin\varphi\,d\xi.$$

In matrix form, equation (6.10) has the form

$$\begin{bmatrix}\eta^{(1)}\\ \vartheta^{(1)}\end{bmatrix}^{\cdot} = \begin{bmatrix}N_{11}N_{12}\\ N_{21}N_{22}\end{bmatrix}\begin{bmatrix}\eta^{(1)}\\ \vartheta^{(1)}\end{bmatrix} + \sum_{i=1}^{4} C_i\,[f_i] + [f]. \tag{6.11}$$

The equation contains four initial parameters.

$$C_1 = \eta(a); \quad C_2 = \eta^{(1)}(a); \quad C_3 = \vartheta(a); \quad C_4 = \vartheta^{(1)}(a).$$

We now consider, as an example, a conical shell (Fig. 24). In this case, we shall have

$$\begin{aligned}
N_{11}\eta^{(1)} &= \frac{E(s)h(s)}{s}\int_a^s \frac{1}{\xi E(\xi)h(\xi)}\int_a^\xi \eta^{(1)}(\xi_1)\,d\xi_1\,d\xi + \\
&+ \frac{\mu(s)}{s}\int_a^s \eta^{(1)}(\xi)\,d\xi - \frac{E(s)h(s)}{s}\int_a^s \frac{\mu(\xi)}{E(\xi)h(\xi)}\eta^{(1)}(\xi)\,d\xi,\\
N_{12}\vartheta^{(1)} &= \frac{E(s)h(s)}{s\tan\alpha}\int_a^s\int_a^\xi \vartheta^{(1)}(\xi_1)\,d\xi_1\,d\xi,\\
N_{21}\eta^{(1)} &= -\frac{\cot\alpha}{sD(s)}\int_a^s\int_a^\xi \eta^{(1)}(\xi_1)\,d\xi_1\,d\xi,\\
N_{22}\vartheta^{(1)} &= \frac{1}{sD(s)}\int_a^s \frac{D(\xi)}{\xi}\int_a^\xi \vartheta^{(1)}(\xi_1)\,d\xi_1\,d\xi + \\
&+ \frac{1}{sD(s)}\int_a^s D(\xi)\,\mu(\xi)\,\vartheta^{(1)}(\xi)\,d\xi - \frac{\mu(s)}{s}\int_a^s \vartheta^{(1)}(\xi)\,d\xi.
\end{aligned} \tag{6.12}$$

If thickness of the conical shell and parameters of elasticity are constant, then equation (6.11) is considerably simplified:

$$N_{11}\eta^{(1)} = \frac{1}{s}\int_a^s \frac{1}{\xi}\int_a^\xi \eta^{(1)}(\xi_1)\,d\xi_1\,d\xi,$$

$$N_{12}\vartheta^{(1)} = \frac{Eh}{\tan\alpha}\,\frac{1}{s}\int\limits_a^s\int\limits_a^{\xi}\vartheta^{(1)}(\xi_1)\,d\xi_1\,d\xi,$$

$$N_{21}\eta^{(1)} = \frac{1}{D\tan\alpha}\,\frac{1}{s}\int\limits_a^s\int\limits_a^{\xi}\eta^{(1)}(\xi_1)\,d\xi_1\,d\xi,$$

$$N_{22}\vartheta^{(1)} = \frac{1}{s}\int\limits_a^s\frac{1}{\xi}\int\limits_a^{\xi}\vartheta^{(1)}(\xi_1)\,d\xi_1\,d\xi,$$

$$f_{11} = \frac{1}{s}\ln\frac{s}{a},\quad f_{12} = \frac{a}{s},$$

$$f_{13} = \frac{Eh}{\tan\alpha}\left(1-\frac{a}{s}\right),\quad f_{14} = 0;$$

$$f_{21} = -\frac{1}{D\tan\alpha}\left(1-\frac{a}{s}\right),\quad f_{22} = 0,\quad f_{23} = \frac{1}{s}\ln\frac{s}{a};\quad f_{24} = \frac{a}{s}.$$

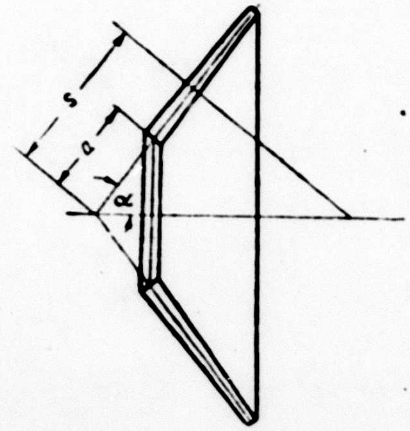

Fig. 24. Conical Shell.

The solution of equation (6.11) is made by the method of subsequent approximations according to scheme indicated in Chapters 3 and 4.

In a number of practical problems, it was found sufficient to use three-four approximations. Another method of approximation, which may be used for solving equations (6.11) -- especially with a gradual convergences of process of successive approximations, -- method of linear approximation.

For the possibility of applying this method, the normal operators must be presented in the form

$$Ny = \sum_{j=1}^{m} Q_j\int\limits_a^s q(\xi)\,d\xi.$$

For operators, entering into equation (6.11), this is readily attained by means of integrating by parts.

Thus, for example, for conical shell we will obtain

$$N_{11}\eta^{(1)} = \frac{E(s)\,h(s)}{s}H(s)\int\limits_a^s\eta^{(1)}(\xi)\,d\xi - \frac{E(s)\,h(s)}{s}\int\limits_a^s H(\xi)\,\eta^{(1)}(\xi)\,d\xi +$$

$$+\frac{\mu(s)}{s}\int\limits_a^s\eta^{(1)}(\xi)\,d\xi - \frac{E(s)\,h(s)}{s}\int\limits_a^s\frac{\mu(\xi)}{E(\xi)\,h(\xi)}\eta^{(1)}(\xi)\,d\xi,$$

$$N_{12}\vartheta^{(1)} = \frac{E(s)\,h(s)}{\tan\alpha}\int\limits_a^s\vartheta^{(1)}(\xi)\,d\xi - \frac{E(s)\,h(s)}{s\tan\alpha}\int\limits_a^s\xi\vartheta^{(1)}(\xi)\,d\xi,$$

$$N_{21}\eta^{(1)} = -\frac{\cot\alpha}{D(s)}\int\limits_a^s\eta^{(1)}(\xi)\,d\xi + \frac{\cot\alpha}{sD(s)}\int\limits_a^s\xi\eta^{(1)}(\xi)\,d\xi,$$

$$N_{ss}\vartheta^{(1)} = \frac{B(s)}{sD(s)}\int_a^s \vartheta^{(1)}(\xi)\,d\xi - \frac{1}{sD(s)}\int_a^s B(\xi)\,\vartheta^{(1)}(\xi)\,d\xi +$$

$$+ \frac{1}{sD(s)}\int_a^s D(\xi)\,\mu(\xi)\,\vartheta^{(1)}(\xi)\,d\xi - \frac{\mu(s)}{s}\int_a^s \vartheta^{(1)}(\xi)\,d\xi,$$

$$H(s) = \int_a^s \frac{d\xi}{\xi E(\xi)\,h(\xi)}; \quad B(s) = \int_a^s \frac{D(\xi)}{\xi}\,d\xi.$$

The method of linear approximation is applied in the form, as discussed in Chap. 3, Sec. 6.

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